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FOR THE CALCULATION OF
ALTERNATING CURRENT PROBLEMS

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FORMULAE AND TABLES

FOR THE CALCULATION OF

ALTERNATING CURRENT PROBLEMS

BY

LOUIS COHEN

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PREFACE.

The importance of mathematics in the study of engineering problems is being generally recognized now and fully appreciated by engineers. The rapid development in alternating current engineering is largely due to the skillful utilization of mathematics in the study of the many complex phenomena which arise in connection with this subject. As a consequence the literature on alternating currents is distinctly mathematical. To be able to utilize fully all the acquired knowledge on this subject, one must have a sufficient mathematical training to enable him to follow the derivations of the many formulae, and be able to interpret them correctly. But even the engineer who has the mathematical equipment for this work is often obliged to consult a good many books and journals before he finds all of the formulae needed for his particular problem.

An attempt has been made here to bring together in a small volume all the important formulae which are necessary and helpful in the solution of alternating current problems. The aim has been throughout to put the formulae in such form that they can be immediately applied for numerical calculations; and in those cases where it seemed desirable, the formulae are illustrated by numerical examples.

In a work of this kind, which is in the nature of a compilation, the author has necessarily drawn freely from all sources of information, but in most cases the formulae were gone over carefully so as to make certain that they are free from errors. Frequently the formulae were derived independently, and in all cases an effort was made to put them in the simplest possible form. A few of the formulae appear here for the first time.

The author is fully aware that accuracy and completeness are of first importance in a book of this character. To this end not only were formulae checked with care, but references to original authorities are generally given. The author will appreciate a notice of errors discovered in the work, and suggestions for additions of other material which might increase the usefulness of the book. Both the scope and arrangement of the present work seem logical to the author, although a great deal of material could have been added, but not without the risk of making the book too cumbersome and expensive for general use.

Many thanks are due to Prof. H. H. Norris of Cornell University for his careful reading of the manuscript and the many valuable suggestions he has made. I am also indebted to my friend Dr. P. G. Agnew of the Bureau of Standards for going over carefully parts of the manuscript and checking some of the numerical examples. The diagrams were prepared by Mr. B. Holland and Mr. A. F. Van Dyck to whom I wish to express my thanks.

I wish also to express my appreciation for the valuable assistance rendered by the publishers and their technical staff in the editing of this book.

LOUIS COHEN.

WASHINGTON, D.C.,
September, 1913.

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ALTERNATING CURRENT PROBLEMS

CHAPTER I

RESISTANCE AND EDDY-CURRENT LOSSES IN METALLIC CONDUCTORS

THE resistance of metallic conductors is higher for alternating currents than for continuous currents due to the so-called skin effect. In the case of alternating currents the distribution of the current is not uniform throughout the entire cross section of the conductor, but tends to concentrate more towards the surface of the conductor, and as a consequence there is a corresponding increase in resistance. The increase in resistance for alternating currents depends on the conductivity and permeability of the conductor and on the frequency. The higher the conductivity, permeability and frequency the greater the increase in resistance.

Besides the I^2R loss, there is another source of energy loss in alternating current circuits which does not occur in continuous current circuits. This is the eddy-current loss in outside conductors. When an alternating current passes through a conductor it creates an alternating magnetic field which will induce currents in conductors placed in that magnetic field and thus causes loss of energy. This is particularly marked in the case of iron bodies placed in a strong alternating magnetic field as in the case, for instance, of a cylindrical iron core inserted within a cylindrical coil. This loss of energy due to eddy currents in outside conducting bodies may be looked upon as an equivalent increase in resistance of the circuit. In fact the outside conductor acts like a closed secondary of a transformer which absorbs energy from the primary.

In this chapter on resistance formulæ we shall consider not only the resistance of conductors proper, but also the increase in effective resistance caused by eddy-current losses in metallic bodies which are placed in magnetic fields.

The general problem of determining the distribution of alternating currents in conductors of any shape offers considerable mathematical difficulties, and has only been worked out for a limited number of special cases. We shall endeavor to give here all the important available formulæ with numerical examples to illustrate their application.

Straight Cylindrical Conductors. — The problem of determining the effective resistance of wires for alternating currents has engaged the attention of many able physicists and several formulæ have been deduced for this case. Owing to the importance of this problem in engineering practice, we shall give here all the formulæ available, with a brief discussion of their limitation and application.

In what follows let

R = resistance for alternating currents.
R_0 = resistance for continuous currents.
$\omega = 2\pi \times$ frequency $= 2\pi n$.
l = length of wire in cm.
μ = permeability.
ρ = specific resistance in C.G.S. units.
d = diameter of wire in cm.

The permeability μ is assumed to be constant.

The form in which the formula for the resistance of cylindrical conductors for alternating currents was first given is as follows:*

$$R = R_0 \left\{ 1 + \frac{1}{12} \frac{\omega^2 l^2 \mu^2}{R_0^2} - \frac{1}{180} \frac{\omega^4 l^4 \mu^4}{R_0^4} + \cdots \right\}. \qquad (1)$$

For non-magnetic materials

$$R = R_0 \left\{ 1 + \frac{1}{12} \frac{\omega^2 l^2}{R_0^2} - \frac{1}{180} \frac{\omega^4 l^4}{R_0^4} + \cdots \right\}. \qquad (2)$$

Formula (1) may also be written in the following form:

$$R = R_0 \left\{ 1 + \frac{n^2 \pi^4 d^4 \mu^2}{48\, \rho^2} - \frac{n^4 \pi^8 d^8 \mu^4}{2880\, \rho^4} + \cdots \right\}. \qquad (3)$$

Formulæ (1) to (3) are applicable only for low frequencies.

When the diameter of the wire is not very small and the frequency fairly high, the series in the above formulæ is very slowly convergent, and it will require more terms in the series for an accurate determination of the resistance.

---

* See J. A. Fleming, "The Principles of Electric Wave Telegraphy and Telephony," second edition, p. 117.

Example. — Copper wire No. 0 B. & S.

$$d = 0.8252 \text{ cm.}, \quad \rho = 1700, \quad n = 500.$$
$$R = R_0 \{1 + 0.081 - 0.0055 + \cdots\} = 1.076\, R_0.$$

This is a rapidly convergent series. The increase in resistance for this case is about 7.6 per cent.

If, however, in the same example we put $n = 1500$ we get

$$R = R_0 \{1 + 0.729 - 0.445 + \cdots\}.$$

The series is very slowly convergent; it will require therefore many more terms to determine the value of R accurately. For still higher frequencies or larger diameter of wire, the series in formulæ (1) to (3) will be still more slowly convergent, and therefore the formulæ will not be suitable. In the case of magnetic material, as that of iron wire, the formulæ is only applicable for very low frequencies.

When the frequency is very high, we have the following convenient formulæ,*

$$\left.\begin{aligned} R &= R_0 \sqrt{\tfrac{1}{2}\, \omega l \mu R_0}, \\ \text{or} \qquad R &= R_0 \sqrt{\frac{\omega \mu \pi\, d^2}{8\, \rho}}. \end{aligned}\right\} \tag{4}$$

For copper wire, $\mu = 1$, $\rho = 1700$ at ordinary temperature, and formula (4) reduces to

$$R \doteq R_0 \frac{\pi d}{82.5} \sqrt{n}. \tag{5}$$

Formulæ (4) and (5) are applicable only when $\dfrac{n \pi^2 d^2 \mu}{\rho}$ has a value much greater than unity, of the order 10 or more.

It is obvious that the above formulæ (1) to (5) have only a limited application. Formulæ (1) to (3) can be used only when $\dfrac{nd^2\mu}{\rho}$ is small, and formulæ (4) and (5) when $\dfrac{nd^2\mu}{\rho}$ is large. Physically it means that the first set of formulæ are applicable to cases in which current traverses almost the entire cross section of the conductor, while formulæ (4) and (5) are applicable to cases in which the current is confined mostly to a thin surface layer of the conductor.

---

* Lord Rayleigh, "The Self-Induction and Resistance of Straight Conductors." *Philosophical Magazine,* Ser. V, May, 1886, Vol. 21, p. 381.

Recently Professor Pedersen* extended the work on this problem so as to cover all cases of current distribution; he also worked out a valuable set of tables which are reproduced here, see Table II, p. 43. The formula is as follows:

$$R = R_0 r\,(ma), \tag{6}$$

$$r\,(ma) = \frac{ma}{2}\,\frac{ber\,(ma)\,bei'\,(ma) - bei\,(ma)\,ber'\,(ma)}{ber'^2\,(ma) + bei'^2\,(ma)}, \tag{7}$$

where $$m = 2\,\pi\sqrt{\frac{2\,n\mu}{\rho}}$$

and $$a = \text{radius of wire.}$$

The value of $r\,(ma)$ for ma from 0 to 6 is given in Table II and for values of ma larger than 6, the expression for $r\,(ma)$ as given in (7) reduces to

$$r\,(ma) = 0.35355\,ma + 0.25 + \frac{0.1326}{ma}. \tag{8}$$

In Table I, p. 42, are given the values of m for copper wire for different frequencies. By the aid of Tables I and II and formulæ (6) and (8) we can compute quite readily the alternating current resistance of wires of any diameter and all frequencies.

To further facilitate the work of computing the resistance of copper wires there are given in Figs. 1 and 2 curves expressing the value $\frac{R}{R_0}$ as a function of $\sqrt{an}$ for values of $\sqrt{an}$ from 1 to 1000.

Example. — Copper wire No. 0 B. & S.,

$$d = 0.8252 \text{ cm.} \qquad \rho = 1700, \qquad n = 500,$$
$$m = 4.824, \qquad ma = 1.99.$$

From Table II we get $r\,(ma) = 1.077$,

hence $$R = 1.077\,R_0,$$

which agrees with result obtained for the same example by formula (3).

If we put in the above example $n = 1500$ we have

$$m = 8.346, \qquad ma = 3.443.$$

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* P. O. Pedersen, *Jahrbuch der Drahtlosen Telegraphie und Telephonie*, Vol. 4, pp. 501–515.

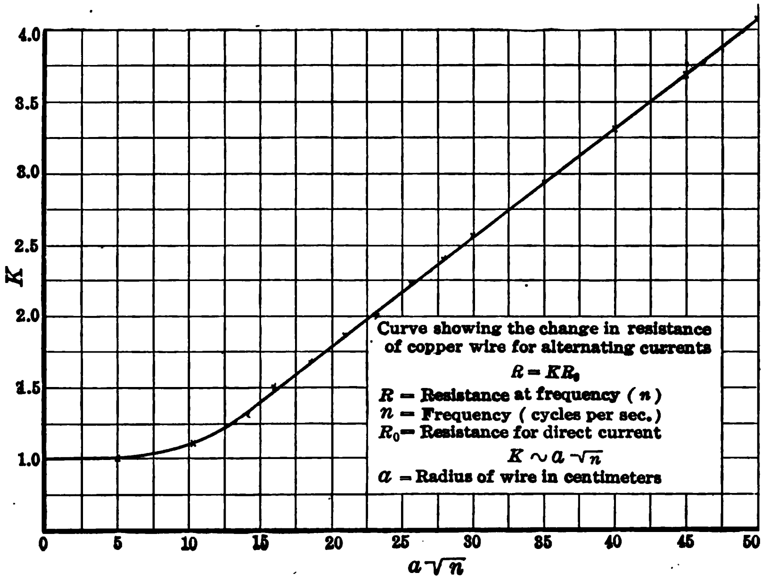
4.0
3.5
3.0
2.5
2.0
1.5
1.0
K
Curve showing the change in resistance of copper wire for alternating currents
R = KR₀
R = Resistance at frequency (n)
n = Frequency (cycles per sec.)
R₀ = Resistance for direct current
K ∿ a√n
a = Radius of wire in centimeters
0
5
10
15
20
25
30
35
40
45
50
a√n

FIG. 1.

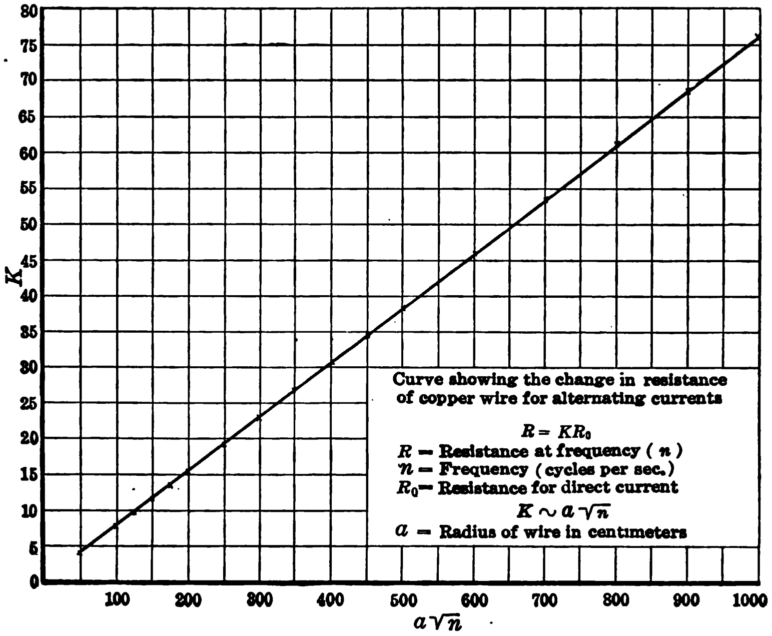
80
75
70
65
60
55
50
45
40
35
30
25
20
15
10
5
0
K
Curve showing the change in resistance of copper wire for alternating currents
R = KR₀
R = Resistance at frequency (n)
n = Frequency (cycles per sec.)
R₀ = Resistance for direct current
K ∿ a√n
a = Radius of wire in centimeters
100
200
300
400
500
600
700
800
900
1000
a√n

FIG. 2.

From Table II, $r\,(ma) = 1.470,$

hence $R = 1.47\,R_0.$

For a frequency 100,000,

$$m = 68.22, \qquad ma = 28.15.$$

Computing $r\,(ma)$ by formula (8) we get

$$r\,(ma) = 10.2,$$

hence $R = 10.2\,R_0.$

Current Penetration in Cylindrical Conductors. — It was pointed out above that the increase in resistance for alternating currents is due to the fact that the current is not uniformly distributed over the entire area of the conductor. The intensity of the current diminishes from the surface to the center of conductor. The amplitude of the current at any point distance r from the axis is given by the formula *

$$I = I_0 \epsilon^{-\sqrt{\frac{2\pi\mu\omega}{\rho}}(a-r)}, \tag{9}$$

where I_0 = current density on surface of conductor.
a = radius of conductor in cm.
ϵ = base of (Naperian) logarithms.

There is also a change in phase in the current from point to point along the radial axis. The intensity of the current diminishes in a geometric progression as the distance from the surface increases in arithmetical progression. For a distance from the surface $a - r = \sqrt{\frac{\rho}{2\pi\mu\omega}}$, the density of the current will be reduced to $\frac{1}{\epsilon} = \frac{1}{2.77}$ its surface value. $\sqrt{\frac{\rho}{2\pi\mu\omega}}$ is taken as the measure of the thickness of current penetration, that is, we may consider the current concentrated in a cylindrical shell of thickness $\left(\frac{\rho}{2\pi\mu\omega}\right)^{\frac{1}{2}}$ and the sectional area of the shell as the effective area for the passage of the current. Denoting by δ the depth of current penetration, that is, the distance from the surface at which the current density is $\frac{1}{\epsilon}$ part of the density at the surface,*

---

* See J. J. Thomson, "Recent Researches in Electricity and Magnetism," p. 281.

$$\delta = \left(\frac{\rho}{2\,\pi\mu\omega}\right)^{\frac{1}{2}}. \tag{10}$$

We have for copper wire ($\rho = 1700$)

$$\delta = \frac{6.5}{\sqrt{n}}$$

for aluminum wire ($\rho = 2800$),

$$\delta = \frac{8.4}{\sqrt{n}}$$

and for iron wire ($\rho = 10{,}000$, $\mu = 1000$)

$$\delta = \frac{0.5}{\sqrt{n}}.$$

Example. — Let $n = 500$.

For copper wire,

$$\delta = \frac{6.5}{\sqrt{500}} = 0.29$$

that is the current penetrates to 0.29 times the radius below the surface of the conductor. Assuming the radius of the conductor to be 1 cm., the total area of the conductor is π sq. cms. The area through which the alternating current is passing is $\pi\,(1^2 - 0.71^2) = 0.496\,\pi$ sq. cm. This is about one-half the actual area and therefore the effective resistance is about double.

By formula (6), we have for this case, $m = 4.824$, $ma = 4.824$, and $r\,(ma) = 1.98$.

The alternating current resistance is 1.98 times the continuous current resistance, which agrees closely with the value obtained above.

Return Conductor. — When conductors are placed close to each other, as in the case of cables, the current distribution is effected by the presence of the return conductor. The greatest current density occurs on the sides of the conductors which face each other. For two conductor cables and low frequencies up to about 2000 cycles per second we have the following formula for the alternating current resistance.*

---

* See "Theorie der Wechselströme" von J. L. La Cour und O. S. Bragstad, Zweite Auflage, p. 568.

$$R=R_0\left\{1+\left[2.02+20.6\left(\frac{d}{2h}\right)^2\right]\left(\frac{nd^2}{\rho}\right)^2-\Big[3.34+267\left(\frac{d}{2h}\right)^2\left(\frac{nd^2}{\rho}\right)^4\right\}, \quad (11)$$

where d = diameter of wire
and h = interaxial distance between wires.

For copper cables formula (11) may be put in the following more convenient form

$$R=R_0\left\{1+\left[0.70+8.5\left(\frac{d}{2h}\right)^2\right]\left(\frac{nd^2}{1000}\right)^2-\left[0.40+32\left(\frac{d}{2h}\right)^2\right]\left(\frac{nd^2}{1000}\right)^4\right\}.$$

For aluminum cables

$$R=R_0\left\{1+\left[0.25+3.0\left(\frac{d}{2h}\right)^2\right]\left(\frac{nd^2}{1000}\right)^2-\left[0.05+4.1\left(\frac{d}{2h}\right)^2\right]\left(\frac{nd^2}{1000}\right)^4\right\}.$$

For magnetic materials, the distance between the wires has very little influence on the current distribution, and we may therefore use the same formula as that for a single conductor.

Example. — Two-conductor copper cable of No. 1 B. & S. wires. Let $n = 300$, $d = 0.73$ cm., $h = 1$ cm.

$$\begin{aligned} R &= R_0\,\{1 + [0.70 + 0.40]\,0.0256 - [0.4 + 4.256]\,0.00065\} \\ &= R_0\,\{1 + 0.028 - 0.0030\} = 1.025\,R_0, \end{aligned}$$

an increase in resistance of about 2.5 per cent.

For a single conductor of the same diameter and the same frequency we have

$$R = R_0\,(1 + 0.018 - \cdots) = 1.018\,R_0,$$

an increase of only 1.8 per cent, hence the presence of the other conductor introduced another increase in resistance of 0.7 per cent.

For very high frequencies the resistance of a two-conductor cable is given by the expression:

$$R=R_0\left\{\frac{\pi h}{\sqrt{h^2-d^2}}\sqrt{\frac{nd^2}{\rho}}+\frac{1}{8}-\frac{\left(h-\sqrt{h^2-d^2}\right)\left(d^2+h\sqrt{h^2-d^2}\right)}{2\,(h^2-d^2)^{\frac{3}{2}}}\right\}. \quad (12)$$

Formula (12) is applicable only when $\frac{2}{\pi d}\sqrt{\frac{\rho}{n}}$ is small compared with unity.

---

* G. Mie, *Wied. Ann.*, 1900. See also "Theorie der Wechselströme," von J. L. La Cour und O. S. Bragstad, *Zweite Auflage*, p. 569.

Example. —

Let $d = 0.5$ cm., $h = 0.75$ cm., $\rho = 1700$ (copper), $n = 10^6$.

$$\frac{2}{\pi d}\sqrt{\frac{\rho}{n}} = 0.05,$$

which is small compared with unity, and hence we may use formula (12) for this case.

$$R = R_0\,(51.1 + 0.125 - 0.335) = 50.9\,R_0.$$

The resistance is about fifty-one times the resistance for continuous currents.

For a single conductor of the same diameter and the same frequency, the resistance is only about twenty times the continuous current resistance.

Resistance of Concentric Conductors. — This problem has been investigated in a very able manner by Dr. Alexander Russell,* who derived a general expression for the resistance and inductance of concentric conductors and from it he deduced several simple formulæ suitable for low frequencies, high frequencies and very high frequencies. The general expression is somewhat unwieldy and not convenient for numerical calculations, and we shall therefore give here only the simplified formulæ for particular cases.

Inner Conductor Solid Metal Cylinder.

Let a = radius of inner conductor in cm.
b = inner radius of concentric outer cylinder in cm.
c = outer radius of concentric outer cylinder in cm.
n = frequency.
ρ = resistivity in C.G.S. units.

$$m = 2\pi\sqrt{\frac{2n\mu}{\rho}}.$$

For direct currents we have

$$R = \frac{\rho}{\pi a^2} + \frac{\rho}{\pi\,(c^2 - b^2)} = R_i + R_0. \qquad (13)$$

The first term in the right hand part of equation (13) is the resistance of inner conductor and the second term the resistance of outer conductor.

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* Alexander Russell, "The Effective Resistance and Inductance of a Concentric Main, and Methods of computing the Ber and Bei and Allied Functions," *Phil. Mag.*, April, 1909.

For low-frequency currents (ordinary commercial frequencies 25 to 60 cycles)

$$R = R_i\left(1 + \frac{m^4a^4}{192}\right) + R_0\left\{1 + m^4(\rho_1 + \rho_2\xi + \rho_3\xi^2)\right\}, \qquad (14)$$

where

$$\rho_1 = \frac{(7c^2 - b^2)(c^2 - b^2)}{192}, \qquad \rho_2 = \frac{b^2c^4}{8(c^2 - b^2)},$$

$$\rho_3 = -\frac{b^2c^6}{4(c^2 - b^2)^2} \quad \text{and} \quad \xi = \log_e\frac{c}{b}.$$

For high-frequency currents, when ma is greater than 5, that is

$$2\pi a\sqrt{\frac{2n}{\rho}} > 5 \quad \text{or} \quad n > \frac{25\rho}{8\pi^2a^2}$$

and for copper, $n > \dfrac{540}{a^2}$,

the following formula is applicable:

$$\left.\begin{aligned} R &= \frac{\rho m}{2\pi a}\left\{\frac{1}{\sqrt{2}} + \frac{1}{2ma} + \frac{3}{8\sqrt{2}m^2a^2}\right\} \\ &\quad + \frac{\rho m}{2\sqrt{2}\pi b}\,\frac{\sinh m(c-b)\sqrt{2} + \sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2} - \cos m(c-b)\sqrt{2}}. \end{aligned}\right\} \qquad (15)$$

The first term gives the resistance of the inner conductor and the second term the resistance of the outer conductor.

When the frequency is very high and c is not nearly equal to b, we have the following expression for the resistance of a concentric conductor:

$$R = \frac{\rho m}{2\pi a\sqrt{2}} + \frac{\rho m}{2\pi b\sqrt{2}} = \sqrt{\mu n\rho}\left(\frac{1}{a} + \frac{1}{b}\right). \qquad (16)$$

Example. — Let the radius of inner conductor, a, be 1 cm. and the external and internal radii of outer conductor 2 and 1.73 cm. respectively. This will give equal sectional areas for the two conductors which is usually the case in practice. If we assume a frequency of 25, we get $m = 2\pi\sqrt{\dfrac{2n}{\rho}} = 2\pi\sqrt{\dfrac{50}{1700}}$ $= 1.08$, and for this value of m with the radii as given in this example, formula (14) is applicable. Introducing these constants in (14),

$m^4 = 1.36,\ \rho_1 = 0.1302,\ \rho_2 = 6,\ \rho_3 = -48,\ \xi = \log_e \frac{2}{\sqrt{3}} = 0.1438$

and

$$R = R_i\left(1 + \frac{1.36}{192}\right)(+ R_0 1 + 1.36 \times 0.0004)$$

$$= R_i\,(1.007) + R_0\,(1.00054).$$

The resistance of inner conductor is increased by 0.7 per cent while the resistance of outer conductor is increased only by 0.05 per cent. The increase in resistance of inner conductor is about fourteen times that of outer conductor.

If in the above example we make $n = 1000$, $m = 2\pi\sqrt{\frac{2n}{\rho}} = 7.4$, ma is greater than 5 and we may therefore use formula (15)

$$\sinh m\,(c - b)\sqrt{2} = 8.2, \qquad \cosh m\,(c - b)\sqrt{2} = 8.3,$$

$$\sin m\,(c - b)\sqrt{2} = 0.332, \qquad \cos m\,(c - b)\sqrt{2} = -0.943,$$

$$R = 1987\,(0.707 + 0.068 + 0.005) + 818 \times \frac{8.2 + 0.332}{8.3 + 0.943}$$

$$= 10^9\,(1550 + 753).$$

For direct current the resistance is $R = 10^9\,(541 + 541)$.

The resistance of inner conductor was increased to about three times the direct current resistance, while the resistance of outer conductor was increased only to one and five-tenths times the direct current resistance.

Concentric Conductors with Hollow Inner Conductor. — If we denote by a_1 and a_2 the internal and external radii of inner conductor, and as before c and b denote the outer and inner radii of outer conductor, we have for low-frequency currents corresponding to formula (14),

$$\left.\begin{aligned} R_i &= \frac{\rho}{\pi\,(a_2^2 - a_1^2)}\left\{1 + m^4\left[\frac{(7\,a_2^2 - a_1^2)\,(a_2^2 - a_1^2)}{192}\right.\right. \\ &\quad \left.\left. + \frac{a_1^2 a_2^4}{8\,(a_2^2 - a_1^2)}\log_e\frac{a_2}{a_1} - \frac{a_1^2 a_2^6}{4\,(a_2^2 - a_1^2)}\left(\log_e\frac{a_2}{a_1}\right)^2\right]\right\}. \\ R_0 &= \frac{\rho}{\pi\,(c^2 - b^2)}\left\{1 + m^4\left[\frac{(7\,c^2 - b^2)\,(c^2 - b^2)}{192}\right.\right. \\ &\quad \left.\left. + \frac{b^2 c^4}{8\,(c^2 - b^2)}\log_e\frac{c}{b} - \frac{b^2 c^6}{4\,(c^2 - b^2)}\left(\log_e\frac{c}{b}\right)^2\right]\right\}. \end{aligned}\right\} \qquad (17)$$

For high frequencies corresponding to formula (16), we have

$$\left.\begin{aligned} R_i &= \frac{\rho m}{2\pi a_1\sqrt{2}}\,\frac{\sinh m(a_2-a_1)\sqrt{2}+\sin m(a_2-a_1)\sqrt{2}}{\cosh m(a_2-a_1)\sqrt{2}-\cos m(a_2-a_1)\sqrt{2}},\\ R_0 &= \frac{\rho m}{2\pi b\sqrt{2}}\,\frac{\sinh m(c-b)\sqrt{2}+\sin m(c-b)\sqrt{2}}{\cosh m(c-b)\sqrt{2}-\cos m(c-b)\sqrt{2}}.\end{aligned}\right\} \quad (18)$$

In the above two formulæ, (17) and (18), R_i is the resistance of inner conductor and R_0 the resistance of outer conductor.

Flat Conductors. — Resistance and distribution of current density in flat conductors with alternating currents.*

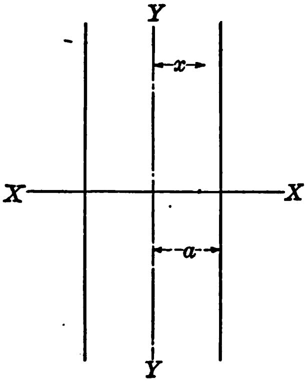

FIG. 3.

Let Fig. 3 represent a section of the conductor, the current flowing in the direction perpendicular to the plane of the paper. We assume that the conductor is very thin compared with its width, as in the case of flat ribbon conductors.

Let $2a$ = thickness of conductor in cm.

ρ = specific resistance in C.G.S. units.

n = frequency.

μ = permeability.

$$m = 2\pi\sqrt{\frac{n\mu}{\rho}}.$$

The current I_x at any point distant x from the center is given by the expression

$$I_x = \frac{E}{\rho}\left\{\frac{\cosh 2mx+\cos 2mx}{\cosh 2ma+\cos 2ma}\right\}^{\frac{1}{2}}. \quad (19)$$

Formula (19) gives only the amplitude of the current at any point in the conductor, but there is also a continuous change in phase of the current from point to point from the surface to the center of the conductor.

The current density at the center of the conductor, $x = 0$, is

$$I_0 = \frac{E}{\rho}\,\frac{\sqrt{2}}{(\cosh 2ma+\cos 2ma)^{\frac{1}{2}}}. \quad (20)$$

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* See Ch. P. Steinmetz, "Theory and Calculation of Transient Electric Phenomena and Oscillations," Chapter VII.

The mean value of the current for the entire section of the conductor is

$$I_m = \frac{E}{\rho ma\sqrt{2}}\left\{\frac{\cosh 2\,ma - \cos 2\,ma}{\cosh 2\,ma + \cos 2\,ma}\right\}^{\frac{1}{2}}. \tag{21}$$

If we denote by R_0 the direct current resistance and by R the effective resistance for alternating currents, we have

$$\frac{R}{R_0} = ma\sqrt{2}\left\{\frac{\cosh 2\,ma + \cos 2\,ma}{\cosh 2\,ma - \cos 2\,ma}\right\}^{\frac{1}{2}}. \tag{22}$$

Special Cases. — If $2\,ma$ is sufficiently large so that we can neglect $\cos 2\,ma$ as compared with $\cosh 2\,ma$, formula (22) reduces to

$$\frac{R}{R_0} = ma\sqrt{2} = 2\pi\sqrt{2}\sqrt{\frac{n\mu}{\rho}}\,a. \tag{23}$$

The increase in resistance is directly proportional to the thickness of plate, the square root of frequency and permeability, and inversely as the square root of the resistivity.

If in formula (19) $2\,ma$ is sufficiently large so that $\cos 2\,ma$ and ϵ^{-2ma} can be neglected compared with ϵ^{2ma}, then for points not far below the surface of the conductor, formula (19) assumes the form

$$I_x = \frac{E}{\rho}\epsilon^{-m(a-x)} = \frac{E}{\rho}\epsilon^{-ms}, \tag{24}$$

where $s = a - x$ is the distance below the surface of conductor. The density of the current diminishes in geometric progression as the distance from the surface increases in arithmetic progression. For the value of $ms = 1$, the current density is reduced to about one-third its maximum value, and from that point on the current density diminishes with great rapidity; hence we can take the value,

$$s = \frac{1}{m} = \frac{1}{2\pi}\sqrt{\frac{\rho}{n\mu}} \tag{25}$$

as the measure of the thickness of current penetration, that is, we may consider the current flowing through only two surface layers of the conductor of thickness s given by formula (25).

We have obtained an identical expression for the case of cylindrical conductors; see formula (10). It is obvious, of course, that in the design of conductors for alternating currents, the mere increase in thickness beyond a certain point will not appreciably decrease the resistance, since the current flow will always be practically limited to two surface layers of thickness $s = \frac{1}{m}\cdot$

Example. — Copper conductor, $\rho = 1700$.

For	$n = 100$,	$s = 0.67$ cm.
	$n = 1000$,	$s = 0.21$ cm.
	$n = 50{,}000$,	$s = 0.03$ cm.

For frequencies, therefore, of 100, 1000 and 50,000 it would be a waste of material to increase the thickness of conductors beyond 1.34 cm., 0.42 cm. and 0.06 cm., respectively. An additional increase in thickness of conductor would diminish the direct current resistance, but it would increase in the same proportion the ratio $\frac{R_0}{R}$ so that the high-frequency resistance would remain sensibly constant.

In the case of iron conductors, even for comparatively low frequencies the current flow is limited to very thin surface layers of the conductors.

Example.—Iron plate, $\rho = 10^4$, $\mu = 10^3$.

For	$n = 50$,	$s = 0.07$ cm.
	$n = 100$,	$s = 0.05$ cm.
	$n = 500$,	$s = 0.022$ cm.
	$n = 1000$,	$s = 0.016$ cm.

Resistance of Coils with Alternating Currents. — The effect of frequency on the increase of resistance of coils is more marked than that of straight conductors. The problem has been treated mathematically by several investigators and formulæ have been derived by which the alternating current resistance of coils can be determined quite accurately.

In formulæ (26) to (35) given below, the following notation is used.

a = radius of coil in cm.
r = radius of wire in cm.
s = number of turns per cm.
ρ = specific resistance in C.G.S. units.
$\omega = 2\pi \times$ frequency $= 2\pi n$.
R_0 = continuous current resistance.
R = alternating current resistance.

A very simple formula, but of very limited application, has been given by Prof. Wien:*

$$\frac{R}{R_0} = 1 + 0.272\left(\frac{2\pi^2 r^2 \omega s}{\rho}\right)^2. \tag{26}$$

This formula is applicable only when the frequency is low or the radius of the conductor is small, that is,

$$8r\sqrt{\frac{rs\pi\omega}{\rho}} < 1.8$$

and
$$s < \frac{1}{2r \times 2.7}.$$

Besides this the coil must be long compared with its diameter. The length of the coil should be greater than seven times its diameter.

A general formula derived by the author,† which is applicable for all frequencies and any size of wire, provided only the windings are close together and the coil is fairly long compared with its diameter, is the following:

$$R = R_0 + \frac{256\, s^2\omega^2\pi r a}{\rho}\sum_{x=1}^{x=\infty}\frac{1}{x^2(\alpha_x^2+\beta_x^2)}\,\frac{\frac{1}{\alpha_x}\sinh 4\alpha_x r + \frac{1}{\beta_x}\sin 4\beta_x r}{\cosh 4\alpha_x r - \cos 4\beta_x r}, \tag{27}$$

where

$$\alpha^2 = \frac{m^2 + \sqrt{m^4 + \frac{16\pi^2\omega^2}{\rho^2}}}{2}, \qquad \beta^2 = \frac{m^2 - \sqrt{m^4 + \frac{16\pi^2\omega^2}{\rho^2}}}{2},$$

$$m = \frac{x\pi}{2r} \quad \text{and} \quad x = 1, 3, 5, 7, \ldots .$$

When r is small (1 or 2 mm.) and the frequency is fairly high then, to a very high degree of approximation,

$$R = R_0 + \frac{256\, s^2\omega^2\pi r a}{\rho}\sum_{x=1}^{x=\infty}\frac{1}{x^2(\alpha_x^2+\beta_x^2)\,\alpha_x}. \tag{28}$$

When the frequency is high and the radius of the wire is not very small, so that we may neglect m^4 compared with $\frac{16\pi^2\omega^2}{\rho^2}$, formula

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* M. Wien, *Ann. d. Physik*, 14, 1, 1904.

† Louis Cohen, *Bulletin Bureau of Standards*, 4, 162, 1907.

(27) reduces to

$$R = R_0 \left\{ 1 + 4\pi r^3 s^2 \sqrt{\frac{\pi\omega}{2\rho}} \right\}. \tag{29}$$

When the frequency is so low that $\frac{16\pi^2\omega^2}{\rho^2}$ may be neglected compared with m^4, formula (27) reduces to

$$R = R_0 \left\{ 1 + \frac{107\, r^6 s^2 \omega^2}{\rho^2} \right\}. \tag{30}$$

Prof. Sommerfeld * derived the following two formulæ corresponding to (29) and (30).

For very high frequencies, when $\frac{r}{2}\sqrt{\frac{2\pi n}{\rho}} > 6$,

$$\frac{R}{R_0} = \frac{r}{2}\sqrt{\frac{2\pi\omega}{\rho}}\, \phi\,(2\pi rs). \tag{31}$$

The value of $\phi\ (2\pi rs)$ has not been determined, but from the experimental results of Black † the following values of ϕ were obtained:

$2\, rs = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9\ \ 1.00$
$\phi = 1,\ 1,\ \ 1.1,\ 1.27,\ 1.45,\ 1.67,\ 1.92,\ 2.2,\ 2.6,\ 3.1,\ 3.7.$

For low frequencies,

$$R = R_0 \left\{ 1 + \frac{1}{4}\frac{\pi^2\omega^2 r^4}{\rho^2}\left[\frac{1}{3} + (2\pi rs)^2 + \frac{1}{216}(2\pi rs)^4 + \frac{1}{518400}(2\pi rs)^8 + \cdots\right]\right\}. \tag{32}$$

Example. —

$r = 0.05$ cm., $\omega = 2\pi \times 5 \times 10^5$, $s = 8$, $\rho = 1700$.

By formula (29) we get

$$\frac{R}{R_0} = 1 + 4\pi\,(0.05)^3 64 \sqrt{\frac{2\pi^2 \times 5 \times 10^5}{2 \times 1700}} = 6.43.$$

By formula (31) we have for $2\, rs = 0.8$, $\phi = 2.6$

$$\frac{R}{R_0} = \frac{0.05}{2} \times 2.6 \sqrt{\frac{4\pi^2 \times 5 \times 10^5}{1700}} = 7.02.$$

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* Sommerfeld, *Ann. d. Phys.*, 24, 609, 1907. See also A. Essau, *Jahrbuch d. Drahtlosen Teleg.*, Vol. 4, 490, 1911.

† Black, *Ann. d. Physik*, 19, 157, 1906.

If in the above example we put $\omega = 2\pi \times 5 \times 10^3$, we may apply formulæ (30) and (32).

By (30)

$$\frac{R}{R_0} = 1 + \frac{107\,(125)^2 \times 10^{-12} \times 64 \times 4\pi^2 \times 25 \times 10^6}{(1700)^2} = 1.037.$$

By (32)

$$\frac{R}{R_0} = 1 + \frac{\frac{1}{4}\pi^2 \times 4\pi^2 \times 25 \times 10^6 \times 625 \times 10^{-8}}{(1700)^2}$$

$$\left\{\frac{1}{3} + (2.51)^2 + \frac{1}{216}(2.51)^4 + \ldots\right\} = 1.036.$$

It is seen that in the above examples the agreement between the two sets of formulæ is fairly good. It may be of interest to compare the difference in the increase in resistance for a coil and straight wire. Using the same data as in the above example, $r = 0.05$ cm. and $n = 5 \times 10^5$, we have by formula (6) for the case of straight wire,

$$\frac{R}{R_0} = r\,(ma), \tag{6'}$$

$$m = 2\pi\sqrt{\frac{2 \times 5 \times 10^5}{1700}} = 132.4, \qquad ma = 7.62,$$

$$r\,(ma) = 0.35355\,ma + 0.25 + \frac{0.1326}{ma} = 2.91,$$

hence $\dfrac{R}{R_0} = 2.91.$

For the same size of wire and the same frequency the wire wound into a coil, $\dfrac{R}{R_0} = 6.43$; hence winding the wire into a coil has more than doubled its high-frequency resistance.

When the frequency is neither very high nor very low, and the windings are close together, the general formula (27) should be used.

Dr. Nicholson * has derived an expression for alternating

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* J. W. Nicholson, *Phil. Mag.*, 19, 77, 1910.

current resistance of coils which is only applicable when the windings are at some distance from each other, as follows:

$$\left.\begin{aligned}\frac{R}{R_0} &= 1 + \frac{1}{12}\frac{r^4\pi^2\omega^2}{\rho^2} - \frac{1}{180}\left(\frac{r^4\pi^2\omega^2}{\rho^2}\right)^2 + \cdots \\ &\quad - \frac{1}{32}\left(\frac{r}{a}\right)^2 \cos^2\alpha \left\{1 - 5\sin^2\alpha 1 + \right\}\left\{\frac{1}{24}\frac{r^4\pi^2\omega^2}{\rho^2} + \cdots\right\},\end{aligned}\right\} \quad (33)$$

where α is the angular pitch of the winding.

Formulæ (26) to (33) are applicable only for single layer coils. For coils of more than one layer we have the following formula given by Wien:

$$\frac{R}{R_0} = 1 + \frac{\pi^4 r^6 m^2}{\rho^2 l^2}\left\{1 - \frac{(r_1 + r_2)^2}{l^2}\right\}\omega^2 \quad (34)$$

where r_1 and r_2 are the inner and outer radii, l the length of the coil and m is the number of layers. This formula holds only for coils of length at least eight times the diameter and when the frequency is not very high, $\left(3\,r\sqrt{\frac{2\,\pi\omega}{\rho}} < 3\right)$.

For flat coils, length of coil small compared with diameter, we have*

$$\frac{R}{R_0} = 1 + \frac{\pi^4 r^6}{\rho^2 (r_1 + r_2)^2}\left\{1 + \frac{3\,r_1^2}{(r_1 + r_2)^2}\right\}\omega^2. \quad (35)$$

Alternating Current Resistance of Large Slot-wound Conductors. — In the case of alternators, induction motors, etc., the conductors are put into iron slots, that is, the conductor is surrounded by iron on three sides. The eddy current losses in such conductors carrying alternating currents are considerable, in some cases many times the I^2R loss. That is, the effective resistance for alternating currents, because of the eddy currents, may be several times the resistance as calculated from the dimensions and specific resistance of the conductor.

This problem was discussed in an interesting paper by Mr. Field,† who obtained a formula giving approximately the effective resistance of the conductor. He also derived expressions for the current distribution, density and phase, in the entire sections of the conductors.

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* A. Essau, *Jahrbuch der Drahtlosen Telegraphie und Telephonie*, Vol. 4, p. 490, 1911.

† A. B. Field, *Proc. A.I.E.E.*, 24, 659, 1905.

If we denote by R_0 the direct current resistance and by R the alternating current resistance, we have for the m^{th} layer

$$K = \frac{R}{R_0} =$$

$$\alpha f \frac{4(m^2 - m)(\cosh \alpha f - \cos \alpha f)(\sinh \alpha f - \sin \alpha f) + (\sinh 2\alpha f - \sin 2\alpha f)}{\cosh 2\alpha f - \cos 2\alpha f}, \tag{36}$$

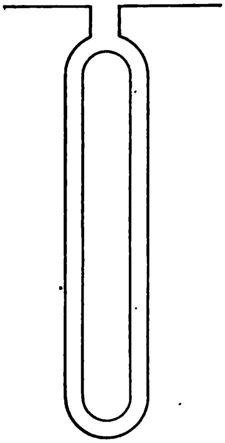

Fig. 4.

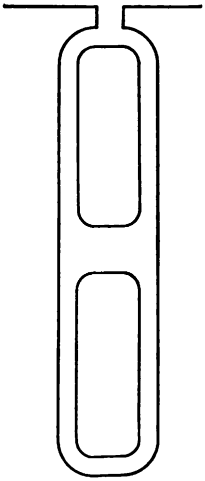

Fig. 5.

where

f = depth of conductor in cm. (if a laminated conductor take the gross depth).

$\alpha = 0.145\sqrt{\frac{nr_1}{r_2}}$.

n = frequency.

r_2 = practically unity for solid conductors, and for laminated conductors is generally equal to the ratio of half the mean length of turn to gross length of core.

r_1 = for solid conductors, the ratio of net copper measured across the slot, to the slot width, and for laminated conductors generally the same multiplied by the ratio of net to gross conductor depth.

m denotes the m^{th} layer, counting the number of layers from the bottom of the slot upwards.

To facilitate the computation of $\frac{R_0}{R}$ as given by equation (36) Mr. Field worked out a set of curves, which are reproduced in Fig. 6, giving the values of $\frac{R}{R_0}$ for different values of αf and m.

As an illustration consider the problem of a two-layer winding having the following dimensions:

Slot width = 0.65 in.

Conductors, 2 – 1 (0.4 × 0.75 in.).

$$r_2 = 1, \quad r_1 = \frac{0.4}{0.65} = 0.615, \quad f = 0.75 \times 2.54 = 1.905.$$

For 25 cycles, $\alpha = 0.145\sqrt{25 \times 0.615} = 0.568, \quad \alpha f = 1.08.$

For 60 cycles, $\alpha = 0.145\sqrt{60 \times 0.615} = 0.88, \quad \alpha f = 1.68.$

From the curves given in Fig. 6 we obtain the following values for $\frac{R}{R_0}$:

25 cycles.

Bottom conductor, $m = 1, \quad \frac{R}{R_0} = 1.12.$

Top conductor, $m = 2, \quad \frac{R}{R_0} = 2.00.$

60 cycles.

Bottom conductor, $m = 1, \quad \frac{R}{R_0} = 1.55.$

Top conductor, $m = 2, \quad \frac{R}{R_0} = 5.6.$

Mr. Field discusses at some length the relative advantages of different form of windings and the original paper should be consulted by those who are particularly interested in this subject.

Distribution of Induction and Eddy Currents in a Long Iron Pipe, Enclosing a Conductor Carrying an Alternating Current. — It is now generally appreciated that single core cables designed to carry alternating currents cannot be laid in separate iron pipes, on account of the large eddy current and hysteresis losses which will be produced by the magnetic induction in the iron pipes. It is

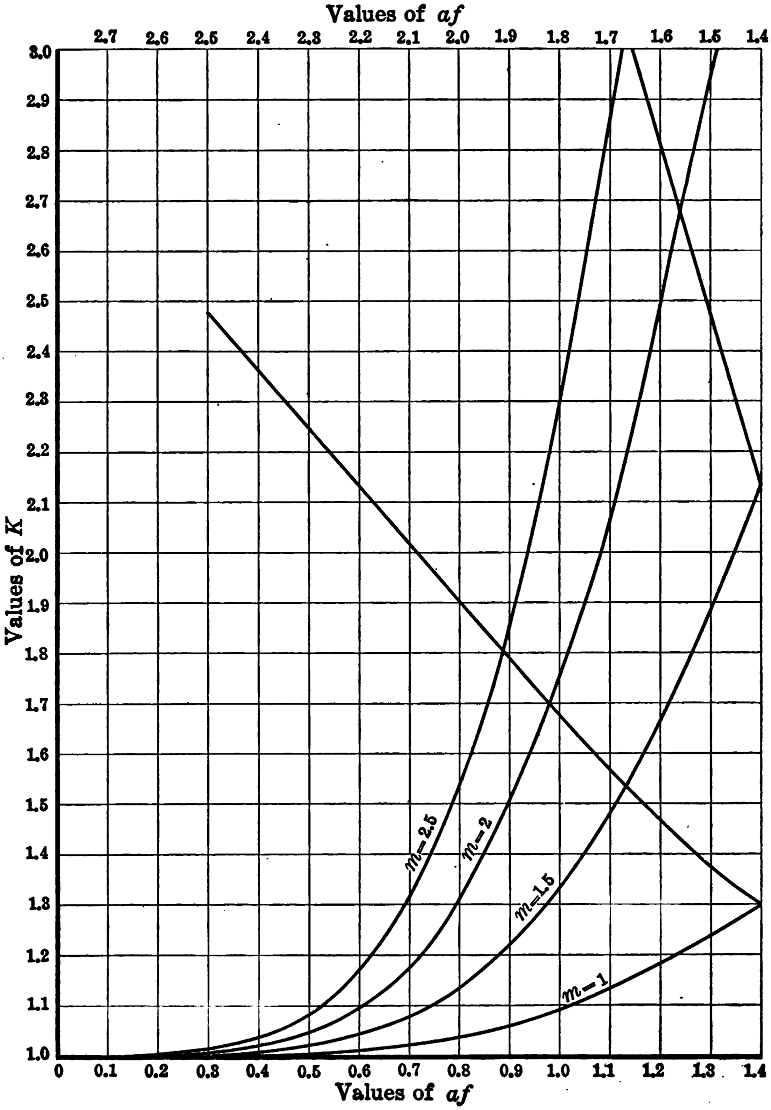
Values of *af*
2.7 2.6 2.5 2.4 2.3 2.2 2.1 2.0 1.9 1.8 1.7 1.6 1.5 1.4
Values of *K*
3.0 2.9 2.8 2.7 2.6 2.5 2.4 2.3 2.2 2.1 2.0 1.9 1.8 1.7 1.6 1.5 1.4 1.3 1.2 1.1 1.0
m=2.5
m=2
m=1.5
m=1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4
Values of *af*

FIG. 6.

interesting, however, to know what the energy loss is likely to be in any particular case, and we give here the formulæ as developed by Mr. Field* for the flux distribution and energy loss in iron pipes enclosing single core cables which carry alternating current.

Let
I = current amplitude in cable,
h = thickness of pipe wall in cm.,
d = internal pipe diameter in cm.,
L = mean circumferential length of pipe in cm.,
μ = mean permeability,
$\eta = \frac{4\pi\mu}{10}$,
$m = 0.0002\sqrt{\frac{\mu n}{\rho}}$,
$\omega = 2\pi \times \text{frequency} = 2\pi n$,
ρ = specific resistance (electrical) iron,
$P = \frac{Im}{L}\sqrt{\frac{\cosh mh - \sin mh}{\cosh mh + \cos mh}}$,
B_x = induction in lines per sq. cm. at point distant x from surface,
I_x = current density in amperes per sq. cm. of the longitudinally flowing current at point distant x from surface,
W = eddy current loss per cm. length of pipe expressed in watts.

Two cases are considered:

(1) Where the pipe is laid in dry concrete or dry earth, so that no eddy currents leak out or leave the pipe.

(2) Where the pipe is laid in wet soil, or where the ends are for any other reason at the same potential.

CASE 1

$$I_x = P\{\epsilon^{mx}\sin(\omega t + mx + \phi) - \epsilon^{m(h-x)}\sin[\omega t + m(h-x) + \phi]\}. \quad (37)$$

The amplitude of I_x may be put in the following form:

$$\bar{I}_x = \frac{I_m}{L}\sqrt{\frac{2\{\cosh m(h-2x) - \cos m(h-2x)\}}{\cosh mh + \cos mh}}. \quad (38)$$

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* M. B. Field, "A theoretical consideration of the currents induced in cable sheaths and the losses occasioned thereby," *Journal Inst. of Elect. Eng.*, Vol. 33, pp. 936–963, Apr., 1904.

For the flux distribution we have

$$B_x = \frac{Pm \times 10^8 \rho}{\omega} \{\epsilon^{mx} (\sin - \cos) (\omega t + mx + \phi)$$
$$+ \epsilon^{m(h-x)} (\sin - \cos)[\omega t + m(h-x) + \phi]\}, \qquad (39)$$

and the amplitude of B_x is

$$\overline{B}_x = \frac{I\eta}{L}\sqrt{\frac{\cosh m(h-2x) + \cos m(h-2x)}{\cosh mh + \cos mh}}. \qquad (40)$$

The energy loss is

$$W = \frac{I^2 m\rho}{L} \frac{\sinh mh - \sin mh}{\cosh mh + \cos mh}. \qquad (41)$$

CASE 2

We have

$$W = \frac{I^2 m\rho}{2L} \frac{\sinh 2mh - \sin 2mh}{\cosh 2mh + \cos 2mh}.$$

$\overline{I}_x$ and $\overline{B}_x$ are of the same form as (38) and (40) except that we write $2h$ in place of h.

Example. —

$d = 2$ cm., $h = 0.1$ cm., $I = 25$ amp., $p = 2\pi \times 60$, $\mu = 1000$,

$\eta = \frac{4\pi\mu}{10} = 1256$, $\rho = 10^{-5}$ ohm, $L = \pi(d+h) = 6.6$ cm.

$m = 15.4$, $mh = 1.54$.

$$\sin 2mh = 0.061, \qquad \cos 2mh = -0.998,$$
$$\sinh 2mh = 10.86, \qquad \cosh 2mh = 10.9,$$

$$W = \frac{625 \times 15.4 \times 10^{-5}}{6.6} \times \frac{10.86 - 0.061}{10.91 - 0.998} = 1590 \times 10^{-5} \text{ watts per cm.}$$

If the pipe is 200 feet long, the energy loss is

$$W = 1590 \times 10^{-5} \times 200 \times 30.5 = 97 \text{ watts.}$$

For 100 amperes under the same conditions the loss is

$$W = 16 \times 97 = 1552 \text{ watts.}$$

Effect of Grounded Lead Covering of Single Conductor Cable. — The lead covering acts as a short-circuited secondary to the e.m.f. generated by the inductive reactance of the line. It reduces therefore the inductance of the line, but increases the energy loss due to the currents in the lead covering.

Let

R = line resistance,

R_1 = resistance of lead covering (for the same cross-section, fourteen times that of copper),

x_0 = line inductive reactance if no lead covering were used and a conductor of the same outside diameter as that of the lead covering were used in the calculations.

The formula obtained by Berg* for the effective resistance and effective reactance are as follows:

$$\left.\begin{aligned} R_{\text{eff}} &= R + \frac{R_1 x_0^2}{R_1^2 + x_0^2}. \\ \text{Effective reactance} &= \frac{R_1^2 x_0}{R_1^2 + x_0^2}. \end{aligned}\right\} \tag{42}$$

As a rule x_0 is small compared with R_1 and we can write,

$$\left.\begin{aligned} \text{Effective resistance} &= R + \frac{x_0^2}{R_1}. \\ \text{Effective reactance} &= x_0. \end{aligned}\right\} \tag{43}$$

To calculate, therefore, the effective resistance and effective reactance of a lead covered cable, we determine the resistances of conductor and resistance lead covering R and R_1, and $x_0 = 2\pi nL$. The inductance L is taken as that between conductors of the same diameter as the outside diameter of the lead covering and the distance, the interaxial distance between the conductor and its return.

Example. — Two No. 0000 B. & S. conductors each enclosed in a lead sheath of external diameter = 1″ and internal diameter = 0.75″. Interaxial distance between the conductors, $D = 4.5''$. Length of conductors = 1000 feet. To determine the effective resistance and reactance at 25 and 60 cycles.

We have $R = 0.049$ ohm,

area of lead $= \frac{1}{8}\pi \times \frac{7}{8} = 0.343$ sq. in.,

and $R_1 = 0.332$ ohm.

At 25 cycles, we have for the inductive reactance,

$$x_0 = 2\pi \times 25\left(2 \log_e \frac{8}{1} + 0.5\right)\frac{1.61 \times 10^5}{10^9} \times \frac{1}{5.28} = 0.0234 \text{ ohm.}$$

---

* Ernst J. Berg, Constants of Cables and Magnetic Conductors, *Proc. A.I.E.E.*, April, 1907.

Therefore,

$$R_{\text{eff}} = 0.049 + \frac{(0.0234)^2}{0.332} = 0.0506 \text{ ohm}.$$

For a frequency of 60 cycles we have

$$x_0 = 0.0561 \text{ ohm}$$

and

$$R_{\text{eff}} = 0.0574 \text{ ohm}.$$

The resistance is increased by 3 per cent for the 25 cycle frequency and by 16 per cent for the 60 cycle frequency.

The table given below applies to a single-conductor cable No. 0000 B. & S. having a lead covering $\frac{1}{16}$-in. thick with an outside diameter of 1 in.

FREQUENCY = 25.

Interaxial distance in inches.	2.13	4	12	24
R	0.264	0.264	0.264	0.264
x_0	0.0497	0.916	0.1544	0.1904
x_0^2	0.0025	0.0084	0.0238	0.0362
R_{eff}	0.2651	0.2676	0.2743	0.2796
Increase	0.42%	1.37%	3.9%	5.9%

FREQUENCY = 60.

Interaxial distance in inches.	2.13	4	12	24
R	0.264	0.264	0.264	0.264
x_0	0.1193	0.220	0.3706	0.4570
x_0^2	0.0142	0.048	0.137	0.208
R_{eff}	0.2701	0.2848	0.3233	0.3536
Increase	2.3%	7.9%	22.5%	33.9%

Effect of Ungrounded Lead Covering in a Single-conductor Cable. — Professor Berg[*] has also derived expressions for the eddy current and energy losses in an ungrounded lead covered cable, as follows:

$$\begin{aligned} &\text{Eddy currents in lead} &&= \frac{\omega L I}{4 R_1}, \\ &\text{Energy loss in lead} &&= \frac{\omega^2 L^2 I^2}{4 R_1}, \\ &\text{Effective resistance, } R_{\text{eff}} &&= R + \frac{\omega^2 L^2}{4 R_1}, \end{aligned} \tag{44}$$

---

[*] l. c., p. 24.

where $\omega = 2\pi \times$ frequency,
R_1 = lead resistance in ohms,
R = resistance of conductor proper in ohms,

$$L = \frac{3.22}{10^4} \log_e \frac{D + \frac{D_1}{4}}{D - \frac{D_1}{4}} \text{ inductance per mile in henrys,}$$

D = interaxial distance between the conductor and its return,
D_1 = external diameter of lead sheath.

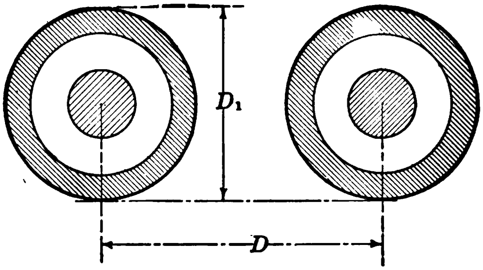

FIG. 7.

As an illustration we may use the same data as given in the example for the case of grounded lead covered cable.

We have then

$$D = 4.5 \text{ in.}, \quad D_1 = 1 \text{ in.},$$

$$\log_e \frac{D + \frac{D_1}{4}}{D - \frac{D_1}{4}} = \log_e \frac{4.75}{4.25} = 0.111,$$

$$L = \frac{3.22}{10^4} \times 0.111 \times \frac{1}{5.28} = \frac{6.8}{10^6} \text{ henry,}$$

$$R = 0.049 \text{ ohm,}$$
$$R_1 = 0.332 \text{ ohm,}$$
$$\omega = 2\pi \times 60,$$

hence,

$$R_{\text{eff}} = 0.049 + \frac{4\pi^2 \times 3600 \times (6.8)^2 \times 10^{-12}}{4 \times 0.332} = 0.049 + 5 \times 10^{-6}.$$

It is seen from this example that the increase in resistance is negligible if the lead covering is not grounded.

Leakage Conductance.*—All insulating materials are more or less conducting to some extent; there is no absolutely perfect insulator. Hence, when a difference of potential exists between two conductors which are embedded in an insulating medium, as in the case of the cores in a cable, there is a current flowing from one conductor to the other through the insulating material separating the two conductors. In consequence of the incomplete insulation and also because of dielectric hysteresis and electrostatic radiation, we have an energy component of the current,

$$I = Eg,$$

which is in phase with the e.m.f.

We give here some formulæ for the calculation of the leakage conductance of linear conductors.

The leakage conductance between a cylindrical conductor and an infinite conducting plane with uniform distance d between the plane and the axis of the cylinder is

$$g = \frac{2\pi l}{\rho \log_e \frac{2d}{a}}, \tag{45}$$

where

a = radius of cylinder in cm.,
d = distance between plane and axis of cylinder in cm.,
l = length of cylinder in cm.,
ρ = specific resistance of the medium in C.G.S. units.

The resistance between a cylinder and an infinite plane is the reciprocal of the conductance given by equation (45).

$$R = \frac{\rho \log_e \frac{2d}{a}}{2\pi l}.$$

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* See "Theorie der Wechselströme," von J. L. La Cour und O. S. Bragstad, Zweite Auflage, pp. 588–596.

Example. —

$$a = 3 \text{ cm.}, \qquad d = 20 \text{ cm.},$$

$$\rho = 5 \times 10^{10}, \qquad l = 100 \text{ cm.},$$

$$\log_e \frac{2d}{a} = \log_e \frac{40}{3} = 2.59,$$

$$R = \frac{5 \times 10^{10} \times 2.59}{2\pi \times 100} = 2 \times 10^{10} \text{ absohms},$$

$$g = \frac{10^{-10}}{2} \text{ absmhos.}$$

If we express ρ in megohms (10^6 ohms) per $\frac{\text{cm.}}{\text{cm.}^2}$ and l in miles, formula (45) reduces to

$$g = \frac{2\pi l \times 1.61 \times 10^5}{10^6 \rho \log_e \frac{2d}{a}}. \tag{46}$$

Prof. Kennelly* derived formulæ for the leakage conductance of linear conductors in terms of antihyperbolic functions. For the formula corresponding to (45), he gives

$$g = \frac{2\pi l}{\rho \cosh^{-1} \frac{d}{a}} = \frac{1.01}{\rho \cosh^{-1} \frac{d}{a}} \text{ mho per mile.} \tag{47}$$

Example. —

$$\frac{d}{a} = 5,$$

$\rho = 5 \times 10^6$ megohms per $\frac{\text{cm.}}{\text{cm.}^2}$, approximately the value for gutta-percha,

$$\log_e \frac{2d}{a} = \log_e 10 = 2.302, \quad \cosh^{-1} \frac{d}{a} = 2.292.$$

By formula (46) we have

$$g = \frac{1.01}{5 \times 10^8 \times 2.30} = 88 \times 10^{-11} \text{ mho per mile.}$$

By formula (47), we get the approximately same result,

$$g = \frac{1.01}{5 \times 10^8 \times 2.29} = 88 \times 10^{-11} \text{ mho per mile.}$$

---

* *Proceedings Am. Phil. Socy.*, Vol. 48, pp. 142–165, 1909.

For two parallel conductors separated by a distance d, and of equal radii a, we have

$$g = \frac{1.01\,l}{2\,\rho \log_e\left(d + \frac{\sqrt{d^2 - 4\,a^2}}{2\,a}\right)} \tag{48}$$

$$= \frac{1.01\,l}{2\,\rho \log_e \frac{d}{a}} \text{ mho, approx.,} \tag{49}$$

or

$$g = \frac{1.01\,l}{2\,\rho \cosh^{-1} \frac{d}{2\,a}}. \tag{50}$$

When $\frac{d}{a}$ is small, formula (49) is considerably in error.

As an example suppose we take $\frac{d}{a} = 2.5$. Then

$$\log_e \frac{d}{a} = 0.916,$$

$$\log_e \frac{d + \sqrt{d^2 - 4\,a^2}}{2\,a} = 0.693,$$

$$\cosh^{-1} \frac{d}{a} = 0.689.$$

From these values it is obvious that (48) and (50) agree closely while (49) is in error about 35 per cent.

For concentric cylinders, we have for the leakage conductance

$$g = \frac{1.01\,l}{\rho \log_e \frac{D}{d}} \text{mho,} \tag{51}$$

where l is the length of cylinders expressed in miles, ρ is specific resistance expressed in megohms, D is the inner diameter of outer cylinder, and d is the outer diameter of inner cylinder.

In the above conductance formulæ it was assumed that the insulation is homogeneous, having the same resistivity for the entire depth of the insulation. If this is not the case the computations are very complex. We may obtain approximate results by assuming that the insulation consists of several layers of different resistivities, and in that case the formula for concentric cylinders is as follows:

$$g = \frac{1.01\,l}{\rho_1 \log_e \frac{d_1}{d} + \rho_2 \log_e \frac{d_2}{d_1} + \cdots \rho_n \log_e \frac{D}{d_n}} \text{ mho,} \qquad (52)$$

where d_x is the diameter of the x^{th} insulation layer.

For a three-core cable we have

$$g = \frac{2.02\,l}{\rho \log_e \left[\frac{d^2}{a^2} \times \frac{(3\,D^2 - 4\,d^2)^3}{(3\,D^2)^3 - (4\,d^2)^3} \right]} \text{ mho,} \qquad (53)$$

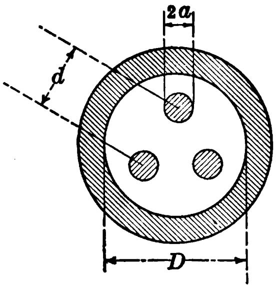

Fig. 8.

where

D = diameter of sheath,

a = radius of each core,

d = interaxial distance between cores.

Magnetic Flux Distribution and Eddy Current Losses in Iron. — It is well known that in subjecting iron to an alternating magnetizing force, eddy currents are generated in the iron in planes perpendicular to the direction of the magnetic flux. These eddy currents produce magnetic fields of their own which prevent the magnetic flux from penetrating completely into the body of the iron. As a consequence we have an uneven distribution of the magnetic flux in the iron and also energy losses. Both of these effects depend on the permeability and conductivity of the iron and on the frequency. The problem of determining the magnetic flux distribution and eddy current losses offers considerable mathematical difficulties, and it has only been worked out for a few special cases which we shall consider here.

Infinite Plates.* — Consider an iron plate, Fig. 9, whose surface in the direction perpendicular to the plane of the paper is infinite in extent, its thickness being 2 a. Assume m.m.f. acting on the plate producing a magnetic field in the direction perpendicular to the plane of the paper, and let $B_a \cos \omega t$ denote the magnetic flux density on the surface of the plate, which is produced by the impressed m.m.f. The magnetic flux density at any point in the direction perpendicular to the surfaces, and at distance x from the center is given by

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* See "Theory and Calculation of Transient Electric Phenomena and Oscillations" (Chapter VI) by C. P. Steinmetz, and "The Theory of Alternating Currents," Vol. I, Chap. XVI, by Alexander Russell.

$$B_x = B_a \sqrt{\frac{\cosh 2mx + \cos 2mx}{\cosh 2ma + \cos 2ma}} \cos(\omega t - \gamma_x), \tag{54}$$

$$\tan\gamma_x = \frac{\sinh m(a-x)\sin m(a+x) + \sinh m(a+x)\sin m(a-x)}{\cosh m(a-x)\cos m(a+x) + \cosh m(a+x)\cos m(a-x)}, \tag{55}$$

where $m = 2\pi\sqrt{\frac{\mu n}{\rho}}$,

μ = permeability,

n = frequency,

ρ = specific resistance in C.G.S. units.

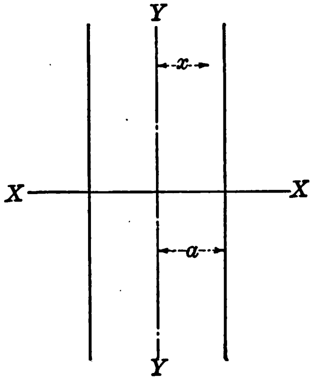

FIG. 9.

From equations (54) and (55) it is obvious that the amplitude of B diminishes from the surfaces to the center of the plate, and the phase of the magnetic flux changes from point to point along the axis perpendicular to the surfaces.

The minimum value of B is at the center of the plate;

$$B_c = \frac{B_a\sqrt{2}}{\sqrt{\cosh 2ma + \cos 2ma}}, \tag{56}$$

$$\tan\gamma_c = \frac{\sinh ma \sin ma}{\cosh ma \cos ma}. \tag{57}$$

The mean or average value of the magnetic flux density for the entire section of the plate is

$$B_m = \frac{B_a}{\sqrt{2}\,am}\left\{\frac{\cosh 2ma - \cos 2ma}{\cosh 2ma + \cos 2ma}\right\}^{\frac{1}{2}}. \tag{58}$$

When $2ma$ is large, equation (58) reduces to

$$B_m = \frac{B_a}{\sqrt{2}\,am}. \tag{59}$$

The total magnetic flux in the plate is

$$B_t = 2\,aB_m = \frac{B_a\sqrt{2}}{m}\left\{\frac{\cosh 2\,ma - \cos 2\,ma}{\cosh 2\,ma + \cos 2\,ma}\right\}^{\frac{1}{2}}. \tag{60}$$

If the flux density had been uniform throughout the entire section of the plate and of an intensity equal to B_a, then the thickness of plate $2\,d$ required to get the same total magnetic flux as that given by equation (60) would have been

$$2\,d = \frac{2}{m\sqrt{2}}\left\{\frac{\cosh 2\,ma - \cos 2\,ma}{\cosh 2\,ma + \cos 2\,ma}\right\}^{\frac{1}{2}}. \tag{61}$$

The value of d as given by equation (61) may be considered as the equivalent depth of uniform magnetic flux density.

When $2\,ma$ is large, equation (61) reduces to

$$d = \frac{1}{m\sqrt{2}} = \frac{1}{2\pi}\sqrt{\frac{\rho}{2\,\mu n}}. \tag{62}$$

Example. —

$$a = 0.025 \text{ cm.}, \quad \mu = 2000, \quad \rho = 10^4, \quad n = 60.$$

$$m = 2\pi\sqrt{\frac{2 \times 10^3 \times 60}{10^4}} = 21.74.$$

x.	cosh $2\,mx$.	cos $2\,mx$.	$\frac{B_x}{B_a}$.
0.000	1.000	1.000	0.972
0.005	1.023	0.976	0.973
0.020	1.403	0.644	0.985
0.025	1.651	0.465	1.000

The equivalent depth, $d = 0.0244$ cm.

For such fine lamination and low frequency, the penetration of magnetic flux density is practically complete. Suppose, however, that in the above example we assume a frequency of 500; then,

$$m = 62.83.$$

x.	cosh 2 mx.	cos 2 mx.	$\frac{B_x}{B_a}$
0.000	1.000	1.000	0.43
0.005	1.205	0.809	0.44
0.010	1.896	0.309	0.46
0.015	3.368	−0.309	0.54
0.020	6.230	−0.809	0.72
0.025	11.590	−1.000	1.00

The equivalent depth d is 0.012 cm. which is about one-half the actual thickness of plate.

The energy loss due to the eddy currents is given by the expression

$$W = \frac{m\rho H_0^2}{32\pi^2 a} \times 10^{-7} \left\{ \frac{\sinh 2\,ma - \sin 2\,ma}{\cosh 2\,ma + \cos 2\,ma} \right\}^{\frac{1}{2}} \frac{\text{watts}}{\text{cu. cm.}} \qquad (63)$$

When ma is greater than π, that is $2\sqrt{\frac{\mu n}{\rho}}\,a$ greater than unity, the factor in the brackets of equation (63) is practically unity, and

$$W = \frac{m\rho H_0^2 \times 10^{-7}}{32\pi^2 a} = \frac{\sqrt{\mu n\rho}}{16\pi a} H_0^2 \times 10^{-7} \frac{\text{watts}}{\text{cu. cm.}}, \qquad (64)$$

H_0 being the magnetizing force.

When ma is small the expression for W as given by (63) is approximately as follows:

$$W = \frac{m\rho H_0^2 \times 10^{-7}}{32\pi^2 a} \times \frac{\frac{1}{6}(2\,ma)^3}{1 + \frac{1}{24}(2\,ma)^4}$$

$$= \frac{m^4\rho^2 H_0^2 \times 10^{-7}}{24\pi^2} \frac{\text{watts}}{\text{cu. cm.}} \text{ approximately.} \qquad (65)$$

For small values of ma the magnetic flux is practically constant through the entire section of the iron plate, and hence we may assume constant permeability. Hence putting $B_{\max} = \mu H_0$, equation (65) becomes

$$W = \frac{1.64\,(2\,a)^2}{\rho} n^2 B_{\max}{}^2 10^{-7} \frac{\text{watts}}{\text{cu. cm.}}. \qquad (66)$$

In the above equation the thickness of the plate $2\,a$ must be expressed in cm. ρ is the resistivity in C.G.S. units (about 10,000), n is the frequency and B is the maximum value of the flux density.

The values of equivalent depths d (of uniform magnetization) in cm. for various materials and frequencies are given below:*

Frequency.	25	60	1000	10,000	10^6
Soft iron, $\mu = 10^3$, $\rho = 10^4$..........	0.0714	0.0460	0.0113	0.0036	0.00036
Cast iron, $\mu = 200$, $\rho = 10^5$..........	0.504	0.325	0.080	0.0252	0.0025
Copper, $\mu = 1$, $\rho = \frac{10^4}{6}$..........	0.922	0.595	0.144	0.0461	0.0046
Resistance alloys, $\mu = 1$, $\rho = 10^5$ ...	7.14	4.60	1.13	0.357	0.036

Iron Cylinders. — The problem of magnetic flux distribution and eddy current losses in iron cylinders has been discussed by several writers, and the results have been summarized in a very

APPROXIMATE FORMULÆ

Function.	$x < 1.$	$x > 6.$
$ber\,(x)$	$1 - \frac{x^4}{64} = 1 - 0.015625\,x^4$	$\frac{e^\alpha}{\sqrt{2\pi x}} \cos\beta\,;\ \alpha = 0.7071\,x$ $+\frac{0.0884}{x} - \frac{0.046}{x^2}$
$bei\,(x)$	$\frac{x^2}{4}\left(1 - \frac{x^4}{576}\right) = \frac{x^2}{4}\,(1 - 0.001736\,x^4)$	$\frac{e^\alpha}{\sqrt{2\pi x}} \sin\beta;\ \beta = 0.7071\,x - 0.3927$ $-\frac{0.0884}{x} - \frac{0.0625}{x^2} - \frac{0.046}{x^3}$
$X\,(x)$	$1 + \frac{x^4}{32} = 1 + 0.03125\,x^4$	$\frac{e^{2\alpha}}{2\pi x}$
$Y\,(x)$	$\frac{x^2}{4}\left(1 + \frac{x^4}{192}\right) = \frac{x^2}{4}\,(1 + 0.00521\,x^4)$	$\overline{X}\left(1 - \frac{0.7071}{x} + \frac{0.25}{x^2} + \frac{0.265}{x^3}\right)$
$Z\,(x)$	$\frac{x^3}{16}\left(1 + \frac{x^4}{384}\right) = \frac{x^3}{16}\,(1 + 0.0026\,x^4)$	$\overline{X}\left(0.7071 - \frac{0.5}{x} - \frac{0.088}{x^2}\right)$
$W\,(x)$	$\frac{x}{2}\left(1 + \frac{x^4}{96}\right) = \frac{x}{2}\,(1 + 0.0104\,x^4)$	$\overline{X}\left(0.7071 + \frac{0.088}{x^2} + \frac{0.125}{x^3}\right)$
$\phi\,(x)$	$1 - \frac{5}{384}x^4 = 1 - 0.01302\,x^4$	$\frac{2}{x} - \frac{0.7071}{x^2} + \frac{0.25}{x^3}$
$\delta\,(x)$	$\frac{x^2}{8}\left(1 - \frac{x^4}{128}\right) = \frac{x^2}{8}\,(1 - 0.00781\,x^4)$	$1 - \frac{0.7071}{x} - \frac{0.25}{x^2}$
$w(x)$	$\frac{x^2}{4}\left(1 - \frac{11}{384}x^4\right) = \frac{x^2}{4}(1 - 0.02865\,x^4)$	$\frac{2.8284}{x} - \frac{2}{x^2} - \frac{0.3536}{x^3}$
$S\,(x)$	$1 - \frac{x^4}{48} = 1 - 0.02083\,x^4$	$\frac{1.4142}{x} + \frac{0.1768}{x^3}$

---

* This table is taken from Steinmetz "Trans. Electrical Phenom.," p. 365.

interesting paper by Prof. P. O. Pedersen, which appeared in the *Jahrbuch der Drahtlosen Telegraphie und Telephonie,* Vol. 4, pp. 501–515. He also worked out very valuable tables to facilitate numerical computations. The formulæ and tables given in this section are taken from that paper.

In the formulæ given in this section, certain functions are employed which are extensions of the *ber* and *bei* functions first introduced by Dr. O. Heaviside and Lord Kelvin. The values of all these functions for arguments from 0 to 6 are given in Table II, p. 43. For arguments greater than 6 or less than unity the values of the functions are given by the foregoing approximate expressions, the largest error in these approximate formulæ being less than 1 per cent.

Magnetic Flux Distribution in Iron Cylinders. — Assume an iron cylinder subjected to an uniform alternating m.m.f. in the direction parallel to the axis of the cylinder. This can be realized in practice by placing the iron cylinder in a long coil through which alternating current is flowing.

Let a = radius of cylinder in cm.,
A = sectional area of cylinder in sq. cm.,
μ = permeability,
ρ = specific resistance in C.G.S. units,
n = frequency,
$\omega = 2\pi n$,
$m = 2\pi\sqrt{\dfrac{2n\mu}{\rho}}$,
$I_0 \cos \omega t$ = current in coil,
N = number of turns per cm.,
$H_a = \dfrac{4\pi N}{10} I_0$ (magnetizing force), the intensity of magnetic field at the surface of cylinder.

The intensity of the magnetic field at any point distant r from the axis of the cylinder is given by

$$H_r = H_r \cos(\omega t - \gamma_r), \tag{67}$$

where

$$H_r = H_a \sqrt{\frac{X(mr)}{X(ma)}},$$

$$\tan \gamma_r = \frac{ber\,(mr)\, bei\,(ma) - bei\,(mr)\, ber\,(ma)}{ber\,(mr)\, ber\,(ma) + bei\,(mr)\, bei\,(ma)}.$$

At the axis of the cylinder

$$\left.\begin{aligned} H_0 &= \frac{H_a}{\sqrt{X\,(ma)}}, \\ \tan\gamma_0 &= \frac{bei\,(ma)}{ber\,(ma)}. \end{aligned}\right\} \tag{68}$$

The total magnetic flux through the cylinder is given by

$$\Phi = \Phi_0 \cos(\omega t - \delta), \tag{69}$$

where and

$$\left.\begin{aligned} \Phi_0 &= A\mu H_a \phi\,(ma) \\ \tan\delta &= \delta\,(ma). \end{aligned}\right\} \tag{70}$$

The density of the eddy currents at any point distant r from the axis of the cylinder is given by

$$J_r = I_r \cos(\omega t - \beta_r), \tag{71}$$

where and

$$\left.\begin{aligned} I_r &= \frac{10}{4\pi} m H_a \sqrt{\frac{Y\,(mr)}{X\,(ma)}} \\ \tan\beta_r &= \frac{ber'\,(mr)\,bei\,(ma) - bei'\,(mr)\,ber\,(ma)}{ber'\,(mr)\,ber\,(ma) + bei'\,(mr)\,bei\,(ma)}. \end{aligned}\right\} \tag{72}$$

At the axis of the cylinder

$$I_0 = 0 \quad \text{and} \quad \tan\beta_0 = -\frac{ber\,(ma)}{bei\,(ma)}. \tag{73}$$

The energy loss due to the eddy currents in one centimeter length of cylinder is

$$W = \frac{\mu n}{8 \times 10^9} A H_a^2 w\,(ma) \text{ watts.} \tag{74}$$

The increase in resistance of the coil due to the dissipation of energy in the iron cylinder or core in one centimeter length of cylinder is

$$R_\omega = \frac{2\pi\mu\omega}{10^9} A N^2 w\,(ma) \text{ ohms.} \tag{75}$$

The inductance per centimeter length of coil is

$$L_\omega = \frac{4\pi}{10^9} \mu A N^2 S\,(ma) \text{ henry.} \tag{76}$$

The increase in inductance of coil per centimeter length due to presence of iron cylinder within the coil is given by

$$\Delta L = L_\omega - \frac{4\pi}{10^9} N^2 A. \tag{77}$$

Equations (67) and (77) give all the necessary formulæ for a complete study of magnetic flux distribution, energy losses, etc., in an iron cylinder placed in a variable magnetic field whose direction is parallel to the axis of the cylinder.

We shall now consider a few examples in illustration of the above formulæ.

Example. — An iron cylinder is subjected to an alternating magnetizing force of intensity H_a in the direction parallel to its axis. What is the intensity of magnetizing force and density of eddy currents at any point along its radius?

Let $a = 0.5$ cm.,
$\mu = 10^3$ (assumed constant for entire section of cylinder),
$\rho = 10^4$ C.G.S. units,
$n = 25$ cycles per second.
We have $m = 14$,
$ma = 7, \quad \sqrt{X\,(ma)} = 22.4.$

r.	$\sqrt{X\,(mr)}$.	$\sqrt{Y\,(mr)}$.	$\frac{H_r}{H_a}$.	$\frac{I_r}{H_a}$.
0.50	22.4	21.28	1	10.58
0.45	13.86	13.10	0.62	6.51
0.40	8.98	8.44	0.40	4.19
0.35	5.90	5.50	0.26	2.73
0.30	3.87	3.57	0.17	1.77
0.20	1.75	1.61	0.08	0.70
0.10	1.06	0.71	0.047	0.35
0.00	1.00	0.00	0.044	0.00

From an inspection of the fourth column in the above table it is seen that for an iron cylinder of 0.5 cm. diameter even for such low frequency as 25 cycles, the magnetization diminishes very rapidly as we recede from the surface of the cylinder. This shows that even for low frequencies lamination is required to obtain complete penetration and not make the eddy currents excessive.

By formula (70) the total magnetic flux in the cylinder is

$$\Phi_0 = \pi\,(0.25)^2 \times 10^3 H_a \phi\,(ma),$$
$$\phi\,(ma) = 0.272,$$
$$\Phi_0 = 53.3\,H_a.$$

For zero frequency $\phi\,(ma) = 1$; hence the effect of the frequency which prevents complete penetration is to reduce the flux to about 27 per cent of its value for steady magnetization.

Referring to formula (70), it is evident that if there were no iron present, the total flux would be $\Phi = AH_a$; hence we may consider the factor $\mu\phi\,(ma)$ as the equivalent permeability of the iron for the variable magnetizing force. Now it may happen that when the frequency is high and the radius of the cylinder is not very small the quantity $\mu\phi\,(ma)$ would be less than unity, which would give the effect of permeability less than unity. This would mean that the penetration is limited to such thin surface layers of the cylinder that the total flux is less than what it would be if the iron were absent and the flux were passing through the air space occupied by the cylinder. For instance in an iron cylinder of 1 cm. radius subjected to alternating magnetizing force, assuming $\mu = 10^3$, $\rho = 10^4$, we have for

$n =$	0	100	10^3	10^4	5.10^4	10^5	10^6
$\mu\phi\,(ma) =$	1000	70	22	7	3.2	2.2	0.7

In this particular example at the frequency of one million cycles per second the total flux is less than if the iron were entirely removed.

Let us now consider a more practical case — that of a solenoidal coil having a core made up of a bundle of iron wires. We shall find the total flux in the core, the energy loss and the increase in resistance of coil due to energy dissipation in the iron core.

We shall assume a coil 10 cm. in diameter and 15 cm. long, total number of turns 60, or 4 turns per centimeter, the frequency 60 cycles per second and the current in the coil 2 amperes. The core is made up of 1000 iron wires, insulated from each other, the radius of each wire being 0.1 cm. We shall also assume a constant permeability of 1000 for the entire section of each wire.

The magnetizing force

$$H_a = \frac{4\pi \times 4}{10} \times 2 = 10.05,$$

$$m = 2\pi\sqrt{\frac{2 \times 60 \times 10^3}{10^4}} = 21.74.$$

The radius of each wire being 0.1 cm.,

$$ma = 2.174.$$

From Table II, page 43, we find

$$\phi\,(ma) = 0.8062,$$
$$w\,(ma) = 0.728.$$

The flux for one wire of the core is

$$\Phi_0 = \pi\,(0.1)^2 \times 10^3 \times 10.05 = 254.5$$

The total flux through the entire core is

$$\Phi = 254.5 \times 1000 = 254{,}500.$$

The energy loss due to eddy currents in one wire is, by formula (74),

$$\frac{10^3 \times 60}{8 \times 10^7}\,\pi\,(10.05)^2 \times 0.728 \times 15 = 0.026 \text{ watt.}$$

The loss of energy in the entire core is

$$W = 0.026 \times 10^3 = 26 \text{ watts.}$$

The increase in resistance of coil due to eddy current losses in one wire is, by formula (75),

$$R_\omega = \frac{2\,\pi \times 10^3 \times 2\,\pi \times 60}{10^9}\,\pi(0.1)^2 \times 16 \times 0.728 \times 15 = 0.013 \text{ ohm.}$$

The increase in resistance due to eddy current losses in the entire core is

$$R_\omega = 0.013 \times 10^3 = 13 \text{ ohms.}$$

Toroidal Coils. — Magnetic flux distribution in an iron core of a toroidal coil.

In the derivation of the formulæ given below it is assumed that the core is made up of a number of very thin plates, insulated from each other, the thickness of the plates being very small compared with the other dimensions.*

Let b = external radius of coils in cm.,
a = internal radius of coils in cm.,
$2\,d$ = thickness of plate in cm.,
μ = permeability (assumed constant for entire section of plate),
$\omega = 2\,\pi n$,
$m = 2\,\pi\sqrt{\frac{n\mu}{\rho}}$,
N = number of turns per cm.,
S = number of plates,
I = intensity of current in coil.

The flux in each plate is

$$\Phi_0 = \frac{\sqrt{2}\,N\mu I}{m}\log_\epsilon\frac{b}{a}\sqrt{\frac{\epsilon^{2\,md} + \epsilon^{-2\,md} - 2\cos 2\,md}{\epsilon^{2\,md} + \epsilon^{-2\,md} + 2\cos 2\,md}}. \qquad (78)$$

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* The formulæ for the toroidal coil are taken from lecture notes by Prof. M. I. Pupin and are published here with his permission.

The total flux in the core is

$$\Phi = \frac{2\,SN\mu I}{m} \log_\epsilon \frac{b}{a} \sqrt{\frac{\epsilon^{2\,md} + \epsilon^{-2\,md} - 2\cos 2\,md}{\epsilon^{2\,md} + \epsilon^{-2\,md} + 2\cos 2\,md}}. \tag{79}$$

For direct current, that is, for complete penetration of the magnetic flux in the core, we have

$$\Phi' = 4\,N\mu ISd \log_\epsilon \frac{b}{a} \tag{80}$$

and

$$\frac{\Phi}{\Phi'} = \frac{1}{2\,dm} \sqrt{\frac{\epsilon^{2\,md} + \epsilon^{-2\,md} - 2\cos 2\,md}{\epsilon^{2\,md} + \epsilon^{-2\,md} + 2\cos 2\,md}}. \tag{81}$$

When md is greater than unity, which is generally true in the case of high frequencies,

$$\frac{\Phi}{\Phi'} = \frac{1}{2\,dm}. \tag{82}$$

As an illustration we may take the data of the coil used by Mr. Alexanderson in his experiments on the effect of frequency on the permeability of iron.[*]

$d = 0.0038$ cm., $b = 2.55$ cm., $a = 0.65$ cm., $s = 10$, $N = 20$,
($\mu = 2.25 \times 10^3$, $\rho = 10^4$, assumed.)

n.	m.	$\frac{\Phi}{\Phi'}$, (by equation 82).
20,000	4.2×10^2	0.31
40,000	5.96×10^2	0.22
80,000	8.4×10^2	0.16
120,000	10.3×10^2	0.13
160,000	11.9×10^2	0.11
200,000	13.3×10^2	0.10

These values agree very closely with the experimental values obtained by Mr. Alexanderson as seen from the curves plotted in Fig. 10. The increase in resistance of the coil due to the eddy current losses in the iron core is given by the formula

$$R = \frac{2\,s\mu N^2\omega}{m} \log_\epsilon \frac{b}{a} \frac{\sinh 2\,md + \sin 2\,md}{\cosh 2\,md + \cos 2\,md}. \tag{83}$$

Using the same data given in the above example,

$d = 0.0038$ cm., $b = 2.55$ cm., $a = 0.65$ cm., $s = 10$, $N = 20$, we have for frequency 20,000 cycles per second

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[*] Magnetic Properties of Iron at Frequencies up to 200,000 Cycles, *P.A.I.E.E.*, Nov., 1911.

$$m = 2\pi\sqrt{\frac{2 \times 10^4 \times 2.25 \times 10^3}{10^4}} = 420,$$

$$2\,md = 3.1,$$

$$\frac{\sinh 2\,md + \sin 2\,md}{\cosh 2\,md + \cos 2\,md} = 1 \text{ approximately,}$$

$$\log_e \frac{b}{a} = \log_e 3.91 = 1.36.$$

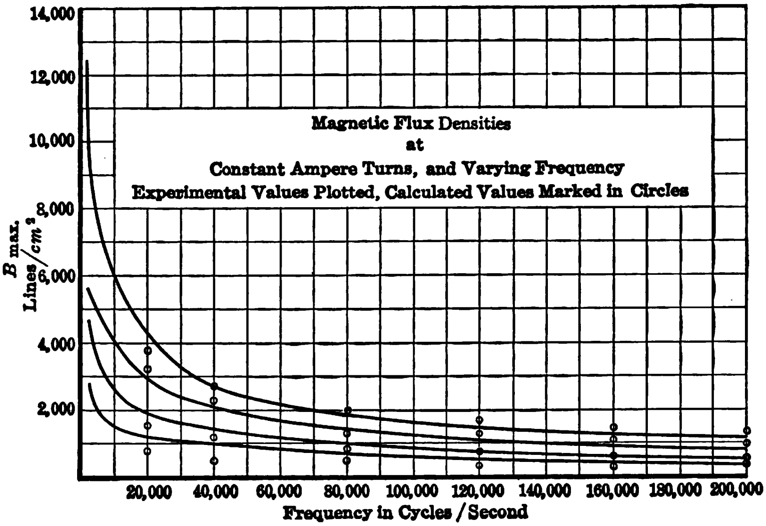

FIG. 10.

Introducing these values in (83) we get

$$R = \frac{2 \times 10 \times 2.25 \times 10^3 \times 400 \times 2\pi \times 2 \times 10^4}{420 \times 10^9} = 7.32 \text{ ohms.}$$

That is, the energy losses in the iron in this particular case are such as would be produced by an additional resistance of 7.3 ohms in the circuit.

TABLE I

Value of m for Copper Conductors

The volume resistivity of high conductivity annealed copper at 60° F. (15.6° C.) is 1696.5,

hence for

$$\mu = 1$$

$$m = 0.2157\sqrt{n}$$

n.	m.	n.	m.
5	0.4824	1,000	6.8221
10	0.6822	2,000	9.6477
15	0.8355	5,000	15.255
20	0.9648	10,000	21.573
25	1.0787	25,000	34.109
30	1.1816	50,000	48.240
35	1.2763	75,000	59.080
40	1.3644	100,000	68.221
45	1.4472	150,000	83.55
50	1.5255	200,000	96.48
60	1.6710	300,000	118.16
75	1.8683	400,000	136.44
100	2.1573	500,000	152.55
200	3.0509	600,000	167.10
300	3.7366	750,000	186.83
400	4.3147	1,000,000	215.73
500	4.8240	2,000,000	305.09

TABLE II

(For use in Formulæ (7) and (67) to (77))

x.	ϕ (x).		S (x).		δ (x).		w (x).		r (x).		l (x).		l_1 (x).		x.
0.00	1	0000	1	0000	0	0000	0	0000	1	000	0	0000	0	5000	0.00
0.05	1	0000	1	0000	0	0003	0	0006	1	000	0	0003	0	5000	0.05
0.10	1	0000	1	0000	0	0013	0	0025	1	000	0	0013	0	5000	0.10
0.15	1	0000	1	0000	0	0028	0	0056	1	000	0	0028	0	5000	0.15
0.20	1	0000	1	0000	0	0050	0	0100	1	000	0	0050	0	5000	0.20
0.25	0	9999	0	9999	0	0078	0	0156	1	000	0	0078	0	5000	0.25
0.30	0	9999	0	9998	0	0113	0	0225	1	000	0	0113	0	5000	0.30
0.35	0	9998	0	9997	0	0153	0	0306	1	000	0	0153	0	5000	0.35
0.40	0	9997	0	9995	0	0200	0	0400	1	000	0	0200	0	5000	0.40
0.45	0	9995	0	9992	0	0253	0	0506	1	000	0	0253	0	5000	0.45
0.50	0	9992	0	9987	0	0312	0	0624	1	000	0	0312	0	4999	0.50
0.55	0	9988	0	9981	0	0378	0	0754	1	001	0	0378	0	4999	0.55
0.60	0	9983	0	9973	0	0450	0	0897	1	001	0	0450	0	4998	0.60
0.65	0	9977	0	9963	0	0527	0	1051	1	001	0	0528	0	4998	0.65
0.70	0	9969	0	9950	0	0611	0	1217	1	001	0	0612	0	4997	0.70
0.75	0	9959	0	9935	0	0701	0	1394	1	002	0	0703	0	4996	0.75
0.80	0	9947	0	9916	0	0798	0	1581	1	002	0	0799	0	4995	0.80
0.85	0	9933	0	9893	0	0900	0	1780	1	003	0	0902	0	4993	0.85
0.90	0	9916	0	9866	0	1007	0	1988	1	003	0	1011	0	4991	0.90
0.95	0	9896	0	9834	0	1121	0	2205	1	004	0	1126	0	4989	0.95
1.00	0	9873	0	9798	0	1240	0	2430	1	005	0	1247	0	4987	1.00
1.05	0	9846	0	9756	0	1365	0	2664	1	006	0	1374	0	4984	1.05
1.10	0	9816	0	9708	0	1495	0	2903	1	008	0	1507	0	4981	1.10
1.15	0	9781	0	9654	0	1631	0	3149	1	009	0	1646	0	4977	1.15
1.20	0	9742	0	9593	0	1771	0	3399	1	011	0	1790	0	4973	1.20
1.25	0	9699	0	9526	0	1917	0	3652	1	013	0	1941	0	4969	1.25
1.30	0	9651	0	9451	0	2067	0	3906	1	015	0	2097	0	4963	1.30
1.35	0	9598	0	9370	0	2221	0	4162	1	017	0	2259	0	4957	1.35
1.40	0	9541	0	9282	0	2379	0	4416	1	020	0	2426	0	4951	1.40
1.45	0	9478	0	9186	0	2541	0	4668	1	023	0	2598	0	4944	1.45
1.50	0	9410	0	9083	0	2706	0	4916	1	026	0	2776	0	4936	1.50

TABLE II (*Continued*)

x.	$\phi(x)$.		$S(x)$.		$\delta(x)$.		$w(x)$.		$r(x)$.		$l(x)$.		$l_1(x)$.		x.
1.50	0	9410	0	9083	0	2706	0	4916	1	026	0	2776	0	4936	1.50
1.55	0	9337	0	8973	0	2875	0	5159	1	029	0	2959	0	4927	1.55
1.60	0	9258	0	8857	0	3046	0	5395	1	033	0	3147	0	4917	1.60
1.65	0	9175	0	8734	0	3219	0	5623	1	037	0	3340	0	4907	1.65
1.70	0	9087	0	8605	0	3394	0	5841	1	042	0	3537	0	4895	1.70
1.75	0	8995	0	8471	0	3571	0	6049	1	047	0	3738	0	4883	1.75
1.80	0	8898	0	8332	0	3748	0	6245	1	052	0	3944	0	4870	1.80
1.85	0	8797	0	8188	0	3926	0	6429	1	058	0	4154	0	4855	1.85
1.90	0	8692	0	8041	0	4104	0	6599	1	064	0	4368	0	4840	1.90
1.95	0	8583	0	7891	0	4281	0	6756	1	071	0	4585	0	4823	1.95
2.00	0	8472	0	7738	0	4457	0	6898	1	078	0	4806	0	4806	2.00
2.05	0	8357	0	7583	0	4632	0	7026	1	086	0	5029	0	4787	2.05
2.10	0	8241	0	7428	0	4805	0	7138	1	094	0	5256	0	4767	2.10
2.15	0	8122	0	7272	0	4976	0	7237	1	102	0	5485	0	4746	2.15
2.20	0	8002	0	7116	0	5144	0	7321	1	111	0	5716	0	4724	2.20
2.25	0	7881	0	6961	0	5309	0	7391	1	121	0	5949	0	4701	2.25
2.30	0	7759	0	6808	0	5470	0	7447	1	131	0	6185	0	4676	2.30
2.35	0	7637	0	6656	0	5627	0	7490	1	141	0	6421	0	4651	2.35
2.40	0	7515	0	6507	0	5780	0	7521	1	152	0	6659	0	4624	2.40
2.45	0	7393	0	6360	0	5928	0	7540	1	164	0	6897	0	4596	2.45
2.50	0	7272	0	6216	0	6072	0	7549	1	175	0	7137	0	4567	2.50
2.55	0	7152	0	6076	0	6210	0	7547	1	188	0	7376	0	4537	2.55
2.60	0	7034	0	5939	0	6343	0	7535	1	201	0	7616	0	4506	2.60
2.65	0	6917	0	5807	0	6471	0	7516	1	214	0	7855	0	4474	2.65
2.70	0	6801	0	5678	0	6594	0	7488	1	228	0	8094	0	4441	2.70
2.75	0	6687	0	5553	0	6711	0	7453	1	242	0	8333	0	4407	2.75
2.80	0	6576	0	5432	0	6822	0	7412	1	256	0	8570	0	4373	2.80
2.85	0	6467	0	5316	0	6928	0	7366	1	271	0	8807	0	4337	2.85
2.90	0	6360	0	5203	0	7029	0	7314	1	286	0	9042	0	4301	2.90
2.95	0	6255	0	5095	0	7124	0	7259	1	302	0	9276	0	4264	2.95
3.00	0	6153	0	4990	0	7214	0	7199	1	318	0	9508	0	4226	3.00

TABLE II (*Continued*)

x.	$\phi(x)$.		$S(x)$.		$\delta(x)$.		$w(x)$.		$r(x)$.		$l(x)$.		$l_1(x)$.		x.
3.00	0	6153	0	4990	0	7214	0	7199	1	318	0	951	0	4226	3.00
3.05	0	6053	0	4890	0	7298	0	7137	1	334	0	974	0	4188	3.05
3.10	0	5956	0	4793	0	7378	0	7072	1	351	0	997	0	4149	3.10
3.15	0	6862	0	4700	0	7452	0	7005	1	368	1	019	0	4109	3.15
3.20	0	5770	0	4611	0	7522	0	6937	1	385	1	042	0	4070	3.20
3.25	0	5680	0	4525	0	7588	0	6867	1	402	1	064	0	4030	3.25
3.30	0	5593	0	4443	0	7649	0	6797	1	420	1	086	0	3990	3.30
3.35	0	5409	0	4364	0	7707	0	6726	1	438	1	108	0	3949	3.35
3.40	0	5427	0	4288	0	7760	0	6654	1	456	1	130	0	3909	3.40
3.45	0	5348	0	4215	0	7810	0	6583	1	474	1	151	0	3868	3.45
3.50	0	5270	0	4144	0	7856	0	6512	1	492	1	172	0	3828	3.50
3.55	0	5195	0	4077	0	7900	0	6441	1	510	1	193	0	3787	3.55
3.60	0	5123	0	4012	0	7940	0	6371	1	529	1	214	0	3746	3.60
3.65	0	5052	0	3949	0	7978	0	6301	1	547	1	234	0	3707	3.65
3.70	0	4984	0	3889	0	8013	0	6233	1	566	1	255	0	3666	3.70
3.75	0	4917	0	3831	0	8045	0	6165	1	584	1	275	0	3626	3.75
3.80	0	4853	0	3775	0	8076	0	6098	1	603	1	295	0	3586	3.80
3.85	0	4790	0	3721	0	8105	0	6032	1	622	1	314	0	3547	3.85
3.90	0	4729	0	3669	0	8132	0	5968	1	641	1	334	0	3508	3.90
3.95	0	4670	0	3619	0	8157	0	5904	1	659	1	353	0	3470	3.95
4.00	0	4613	0	3570	0	8181	0	5842	1	678	1	373	0	3432	4.00
4.05	0	4557	0	3523	0	8203	0	5781	1	697	1	392	0	3394	4.05
4.10	0	4503	0	3478	0	8225	0	5721	1	715	1	411	0	3357	4.10
4.15	0	4450	0	3434	0	8245	0	5662	1	734	1	430	0	3310	4.15
4.20	0	4399	0	3391	0	8264	0	5605	1	752	1	448	0	3284	4.20
4.25	0	4349	0	3349	0	8283	0	5548	1	771	1	467	0	3248	4.25
4.30	0	4300	0	3309	0	8301	0	5493	1	789	1	485	0	3213	4.30
4.35	0	4253	0	3270	0	8318	0	5439	1	808	1	504	0	3179	4.35
4.40	0	4207	0	3231	0	8334	0	5386	1	826	1	522	0	3145	4.40
4.45	0	4162	0	3194	0	8350	0	5335	1	845	1	540	0	3111	4.45
4.50	0	4118	0	3158	0	8366	0	5284	1	863	1	558	0	3078	4.50

TABLE II (*Concluded*)

x.	$\phi(x)$.		$S(x)$.		$\delta(x)$.		$w(x)$.		$r(x)$.		$l(x)$.		$l_1(x)$.		x.
4.50	0	4118	0	3158	0	8366	0	5284	1	863	1	558	0	3078	4.50
4.55	0	4075	0	3123	0	8381	0	5235	1	881	1	576	0	3046	4.55
4.60	0	4033	0	3089	0	8396	0	5186	1	899	1	594	0	3014	4.60
4.65	0	3992	0	3055	0	8410	0	5139	1	917	1	612	0	2983	4.65
4.70	0	3952	0	3022	0	8424	0	5092	1	935	1	630	0	2952	4.70
4.75	0	3913	0	2990	0	8438	0	5047	1	953	1	648	0	2922	4.75
4.80	0	3874	0	2959	0	8452	0	5002	1	971	1	666	0	2893	4.80
4.85	0	3837	0	2928	0	8465	0	4958	1	989	1	684	0	2864	4.85
4.90	0	3800	0	2899	0	8479	0	4915	2	007	1	702	0	2835	4.90
4.95	0	3764	0	2869	0	8492	0	4873	2	025	1	720	0	2807	4.95
5.00	0	3729	0	2841	0	8505	0	4832	2	043	1	737	0	2780	5.00
5.05	0	3695	0	2813	0	8518	0	4792	2	060	1	755	0	2753	5.05
5.10	0	3661	0	2785	0	8531	0	4752	2	078	1	773	0	2727	5.10
5.15	0	3628	0	2758	0	8544	0	4713	2	096	1	791	0	2701	5.15
5.20	0	3595	0	2731	0	8557	0	4675	2	114	1	808	0	2675	5.20
5.25	0	3563	0	2706	0	8569	0	4637	2	131	1	826	0	2650	5.25
5.30	0	3532	0	2680	0	8582	0	4600	2	149	1	844	0	2626	5.30
5.35	0	3501	0	2655	0	8594	0	4564	2	166	1	862	0	2602	5.35
5.40	0	3471	0	2631	0	8607	0	4528	2	184	1	880	0	2578	5.40
5.45	0	3441	0	2606	0	8619	0	4493	2	201	1	897	0	2555	5.45
5.50	0	3412	0	2583	0	8631	0	4458	2	219	1	915	0	2532	5.50
5.55	0	3383	0	2559	0	8643	0	4424	2	236	1	933	0	2510	5.55
5.60	0	3355	0	2537	0	8655	0	4391	2	254	1	951	0	2488	5.60
5.65	0	3327	0	2514	0	8667	0	4358	2	271	1	969	0	2467	5.65
5.70	0	3300	0	2492	0	8678	0	4325	2	289	1	986	0	2446	5.70
5.75	0	3273	0	2470	0	8690	0	4294	2	306	2	004	0	2425	5.75
5.80	0	3246	0	2449	0	8701	0	4262	2	324	2	022	0	2404	5.80
5.85	0	3220	0	2428	0	8713	0	4231	2	341	2	040	0	2384	5.85
5.90	0	3195	0	2407	0	8724	0	4200	2	359	2	058	0	2364	5.90
5.95	0	3170	0	2387	0	8735	0	4170	2	376	2	076	0	2345	5.95
6.00	0	3145	0	2367	0	8746	0	4141	2	394	2	093	0	2326	6.00

TABLE III

Specific Resistance of Metallic Wires

Substance.	Resistance at 0° C. of a wire 1 cm. long, 1 sq. cm. in section.	Resistance at 0° C. of a wire 1 mm. in diameter, 1 meter long.	Resistance at 0° C. of a wire 1 meter long, weighing 1 gram.	Resistance at 0° C. of a wire 1 foot long, $\frac{1}{1000}$ inch in diameter.	Resistance at 0° C. of a wire 1 foot long, weighing 1 gram.	Percentage increase of resistance for 1° C. increase of temp. at 20° C.
Silver annealed	1.460 10^{-6}	0.01859	0.1523	8.781	0.2184	0.377
Silver hard drawn	1.585 10^{-6}	0.02019	0.1659	9.538	0.2379	
Copper annealed	1.584 10^{-6}	0.02017	0.1421	9.529	0.2037	0.388
Copper hard drawn	1.619 10^{-6}	0.02062	0.1449	9.741	0.2078	
Gold annealed	2.088 10^{-6}	0.02659	0.4025	12.56	0.5771	0.365
Gold hard drawn	2.125 10^{-6}	0.02706	0.4094	12.78	0.5870	
Aluminium annealed	2.906 10^{-6}	0.03699	0.0747	17.48	0.1071	
Zinc pressed	5.613 10^{-6}	0.07146	0.4012	33.76	0.5753	0.365
Platinum annealed	9.035 10^{-6}	0.1150	1.934	54.35	2.772	
Iron annealed	9.693 10^{-6}	0.1234	0.7551	58.31	1.083	
Nickel annealed	12.43 10^{-6}	0.1583	1.057	74.78	1.515	
Tin pressed	13.18 10^{-6}	0.1678	0.9608	79.29	1.377	0.365
Lead pressed	19.14 10^{-6}	0.2437	2.227	115.1	3.193	0.387
Antimony pressed	35.42 10^{-6}	0.4510	2.379	213.1	3.410	0.389
Bismuth pressed	130.9 10^{-6}	1.667	12.86	787.5	18.43	0.354
Mercury pressed	94.07 10^{-6}	1.198	12.79	565.9	18.34	0.072
Platinum-silver, 2 parts Ag, 1 part Pt, by weight	24.33 10^{-6}	0.3098	2.919	146.4	4.186	0.031
German silver	20.89 10^{-6}	0.2660	1.825	125.7	2.617	0.044
Gold-silver, 2 parts Au, 1 part Ag, by weight	10.84 10^{-6}	0.1380	1.646	65.21	2.359	0.065

CHAPTER II

INDUCTANCE

Inductance. — The subject of inductance may be conveniently divided into three parts: viz., self and mutual inductance of air-core coils, self and mutual inductance of linear conductors, and inductance of iron-core coils. In the case of air-core coils the inductance depends only on the number of turns and the geometrical configuration of the coil. For the same length of wire we may have as many different inductances as there are different shapes in which the coils can be wound. Obviously since the inductance depends upon the shape of the coil, we shall require a large number of formulæ to cover all possible cases. A collection of formulæ for self and mutual inductance of air-core coils by E. B. Rosa and Louis Cohen has been published in the *Bulletin of the Bureau of Standards.** We have attempted to give there a complete list of all available formulæ, and for this reason in many cases we gave several different formulæ for the same case. This may be desirable in extremely accurate work, as it makes it possible to check the calculations by different formulæ and thus discover any possible error.

We shall make use of that paper here in the discussion of self and mutual inductance of air-core coils, but we shall only give those formulæ which are considered of most practical value, and for a more exhaustive treatment of this subject the reader may be referred to that paper. We also give here an extensive list of formulæ for the self and mutual inductance of linear conductors. On account of the practical importance of linear conductors in the study of transmission line problems, the subject is covered quite fully.

Mutual Inductance of Two Coaxial Circles. — There are a large number of formulæ available for the calculation of mutual inductance of two coaxial circles. Some are applicable only for short distances between the circles, others are applicable only for large distances between circles while some are more general. There is one formula which is exact for all possible cases, but that formula involves the use of elliptic integrals which makes the computation laborious.

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* *Bulletin Bureau of Standards*, Vol. 5, pp. 1–132.

We shall give here only three formulæ which will cover all practical cases.

Let a and A denote the radii of the two circles,

c denote difference in the two radii, $c = A - a$,

d denote distance between the circles,

$$r = \sqrt{c^2 + d^2}, \qquad \alpha = \frac{r^2}{Aa}.$$

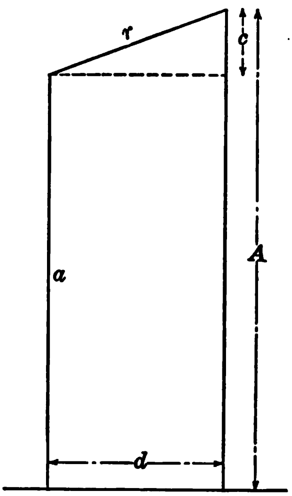

FIG. 11.

The formula for the mutual inductance is

$$M = 4\pi\sqrt{Aa}\left\{\left[1 + \frac{3}{16}\alpha - \frac{15}{1024}\alpha^2 + \frac{35}{(128)^2}\alpha^3 - \frac{1575}{2\,(128)^3}\alpha^4 + \cdots\right]\right.$$
$$\left.\log_e\frac{8\sqrt{Aa}}{r} - \left[2 + \frac{1}{16}\alpha - \frac{31}{2048}\alpha^2 + \frac{247}{6\,(128)^2}\alpha^3 - \frac{7795}{8\,(128)^3}\alpha^4 + \cdots\right]\right\}. \quad (1)$$

This expression gives very accurate results for values of d almost as great as the radius a. When great accuracy is not required this formula may be used even for considerable larger values of d.

A more general formula which is applicable in almost all possible cases with respect to size and distance of circles is the following:

$$M = 4\pi\sqrt{Aa}\cdot 4\pi q^{\frac{3}{2}}\,(1 + \epsilon), \qquad (2)$$

where

$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots,$$

$$l = \frac{1 - \sqrt{k}}{1 + \sqrt{k}}, \qquad k = \frac{2\sqrt{R_1R_2}}{R_1 + R_2},$$

$$R_1 = \sqrt{(A + a)^2 + d^2}, \qquad R_2 = \sqrt{(A - a)^2 + d^2},$$

$$\epsilon = 3\,q^2 - 4\,q^3 + 9\,q^4 - 12\,q^5 + \cdots.$$

ϵ is usually a very small quantity and can be neglected in most cases.

When the circles are of nearly the same radius and the distance between them is small compared with the radius, we have for an approximate value of the mutual inductance the following simple expression:

$$M = 4\pi a\left(\log_e \frac{8a}{r} - 2\right). \tag{3}$$

When only a rough approximation is desired formula (3) may be used even when the circles are at some distance from each other.

Example. —

$$a = 20 \text{ cm.}, \qquad A = 20 \text{ cm.}, \qquad d = 1 \text{ cm.}$$

By formula (1),

$$\log_e \frac{8\sqrt{Aa}}{r} = \log_e 160 = 5.0752,$$

$$\alpha = \frac{1}{400},$$

$$M = 4\pi \times 20\{[1 + 0.00048 - \cdots]5.075 - [2 + 0.00016 - \cdots]\}$$
$$= 773.08 \text{ cm.} = 0.000773 \text{ mh.}$$

By formula (3),

$$M = 80\pi(5.075 - 2) = 772.88 \text{ cm.}$$

For such small distances the error in formula (3) is only one part in a thousand.

Example. —

$$a = 20 \text{ cm.}, \qquad A = 25 \text{ cm.}, \qquad d = 10 \text{ cm.}$$

By formula (1),

$$\log_e \frac{8\sqrt{aA}}{r} = \log \frac{8\sqrt{500}}{\sqrt{125}} = \log 16 = 2.7726,$$

$$\alpha = \frac{r^2}{aA} = \frac{1}{4},$$

$$M = 4\pi\sqrt{500}\{[1 + 0.04687 - 0.00092 + \cdots]2.7726$$
$$- [2 + 0.01562 - 0.00094 + \cdots]\} = 248.815 \text{ cm.}$$
$$= 0.0002488 \text{ mh.}$$

By formula (2) we have

$$R_1 = \sqrt{(45)^2 + 100} = 46.098, \quad R_2 = 11.180,$$
$$k = 0.7927, \quad \sqrt{k} = 0.8903,$$
$$l = 0.05803, \quad q = 0.02901, \quad q^{\frac{3}{2}} = 0.07027, \quad \epsilon = 0.0024,$$
$$M = 16\pi^2\sqrt{500} \times 0.07027 \times 1.0024$$
$$= 248.72 \text{ cm.}$$

By formula (3),

$$\log_e \frac{8a}{r} = \log_e \frac{160}{\sqrt{50}} = 3.1192,$$

$$M = 80\pi\,(3.1192 - 2) = 281.28 \text{ cm.}$$

It is seen that in this case for $d = \frac{a}{2}$, formulæ (1) and (2) give practically identical results, while formula (3) is in error by about 13 per cent.

Attraction Between Circular Currents. — The force of attraction or repulsion between two coaxial circular currents may be expressed as the rate of variation of the mutual inductance with respect to the distance between the circular currents, and is given by the following expression:

$$F = \frac{\partial M}{\partial z} = \frac{-z}{\sqrt{Aa}}\, 192\,\pi^2 q^{\frac{3}{2}}\,(1 + 20\,q^2 + 225\,q^4 + 1840\,q^6 + \cdots), \quad (4)$$

where
$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \cdots,$$

$$l = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}, \quad k' = \frac{\sqrt{(A-a)^2 + z^2}}{\sqrt{(A+a)^2 + z^2}}.$$

a and A are the radii of the two circles and z is the axial distance between them.

To facilitate calculations we give a table at the end of the chapter (Table IV, p. 79) which, together with the formula, is taken from Prof. Nagaoka's paper,* giving the logarithm of $\frac{-\sqrt{Aa}}{z}\frac{\partial M}{\partial z}$ for values of q from 0.020 to 0.150 which are sufficient for most practical cases.

Example. —

$$a = A = 10 \text{ cm.}, \qquad z = 10 \text{ cm.},$$

$$k' = \frac{10}{\sqrt{500}} = \sqrt{\frac{1}{5}} = 0.4472, \quad \sqrt{k'} = 0.6687,$$

$$l = \frac{1 - 0.6687}{1 + 0.6687} = 0.1985,$$

$$q = 0.0992.$$

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* H. Nagaoka, "Attraction between Two Coaxial Circular Currents," *Tokyo Sugaku — Buturigakkwai Kizi,* 2nd Ser., Vol. 6, No. 10.

From Table IV by interpolation we get

$$\log_{10}\left(\frac{-\sqrt{Aa}}{z}\ \frac{\partial M}{\partial z}\right) = 0.85027,$$

therefore
$$\frac{-\sqrt{Aa}}{z}\ \frac{\partial M}{\partial z} = 7.055$$

and
$$\frac{-\partial M}{\partial z} = F = \frac{\sqrt{Aa}}{z} \times 7.055 = 7.055.$$

Mutual Inductance of Two Coaxial Coils. — Let there be two coaxial coils of mean radii A and a, axial breadth of coils, b_1 and b_2 radial depth, c_1 and c_2, and distance apart of their mean planes d. Suppose them uniformly wound with n_1 and n_2 turns of wire. The mutual inductance M_0 between the two central turns of the coils is given by formulæ (1) to (3) and the mutual inductance M of the two coils of n_1 and n_2 turns is then, to a first approximation,

$$M = n_1 n_2 M_0. \tag{5}$$

Formula (5) would be absolutely correct if all the turns were concentrated in the space occupied by the central turns, but owing to the appreciable sections of the coils a correction factor must be applied. The correction term is usually small, not over one per cent in most cases.

In the case of square-sectioned coils we can obtain an accurate determination of M by (5) if in calculating M_0 we use a modified radius r instead of the mean radius a, the value of r being given by the expression

$$r = a\left(1 + \frac{b^2}{24\,a^2}\right) \tag{6}$$

where b is the side of the square section.

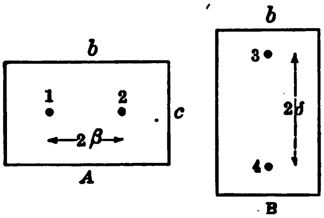

Fig. 12.

If the coil has a rectangular section it can be replaced by two filaments (Fig. 12) the distance apart of the filaments being called the equivalent breadth or the equivalent depth of the coil.

$$\left.\begin{aligned} \beta^2 &= \frac{b^2 - c^2}{12}, \quad 2\,\beta \text{ is the equivalent breadth of } A, \\ \delta^2 &= \frac{c^2 - b^2}{12}, \quad 2\,\delta \text{ is the equivalent depth of } B. \end{aligned}\right\} \tag{7}$$

The equivalent radius of A is given by the same expression which holds for a square coil, viz.,

$$r = a\left(1 + \frac{c^2}{24\,a^2}\right).$$

In the coil B the equivalent filaments have the radii $r + \delta$ and $r - \delta$ respectively, where

$$r = a\left(1 + \frac{b^2}{24\,a^2}\right).$$

The mutual inductance may now be accurately determined. If each has a square section, it is necessary only to calculate the mutual inductance of the two equivalent filaments given by (6). For coils of rectangular sections as A, B, the mutual inductance is the sum of the mutual inductances of the two filaments A on the two filaments on B and M_0 in (5) is the mean of the four inductances M_{13}, M_{14}, M_{23}, M_{24} where M_{13} is the mutual inductance of filament 1 on filament 3, etc.

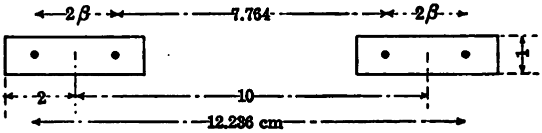

FIG. 13.

Example.—

$$a = A = 25\text{ cm.},\quad b_1 = 4,\quad c_1 = 1,\quad d = 10,$$

$$r = 25\left(1 + \frac{1}{15{,}000}\right) = 25.00167,$$

$$\beta^2 = \frac{b^2 - c^2}{12} = 1.25,\quad 2\beta = 2.236\text{ cm.},$$

the distance apart of the two filaments which replace the coil. We now find by formula (1) or (2) the mutual inductance of the two circles 1, 2, on the two circles 3, 4, where $a = 25.00167$ and $d = 7.764$, 10 and 12.236 cm. respectively. Thus,

$$\begin{aligned} 2\,M_{13} &= 215.00228\,\pi, \\ M_{14} &= 90.31304\,\pi, \\ \frac{M_{23}}{4\,M} &= \frac{130.14060\,\pi}{435.45592\,\pi}, \\ M &= 108.8640\,\pi, \\ \frac{M_0}{\Delta M} &= \frac{107.4885\,\pi}{1.3755\,\pi}. \end{aligned}$$

ΔM = the correction for sections of the coils whose dimensions are given above. The correction factor is of the order of one per cent and, therefore, unless great accuracy is desired, formula (5) will be suitable for the calculation of the mutual inductance of coils.

Mutual Inductance of Concentric Coaxial Solenoids. — If the solenoids are of equal length and we denote by A and a the radii of the outer and inner solenoids respectively, and by n_1 and n_2 the number of turns of wire per centimeter on the single layer winding of the outer and inner solenoids, we have for the mutual inductance

$$M = 4\pi^2 a^2 n_1 n_2 (l - 2A\alpha), \tag{8}$$

where
$$\alpha = \frac{l-r+A}{2A} - \frac{a^2}{16A^2}\left(1-\frac{A^3}{r^3}\right) - \frac{a^4}{64A^4}\left(\frac{1}{2}+\frac{2A^5}{r^5}-\frac{5}{2}\frac{A^7}{r^7}\right) - \frac{35a^6}{2048A^6}\left(\frac{1}{7}-\frac{8A^7}{7r^7}+\frac{4A^9}{r^9}-\frac{3A^{11}}{r^{11}}\right)+\cdots,$$

$$r = \sqrt{A^2+l^2}.$$

Putting $M = M_0 - \Delta M$, $M_0 = 4\pi^2 a^2 n_1 n_2 l$ is the mutual inductance of an infinite outer solenoid and the finite inner solenoid, while ΔM is the correction due to the ends. The expression for α is rapidly convergent when a is considerably smaller than A. Equation (8) shows that the mutual inductance is proportional to $l - 2A\alpha$, that is, the length l must be reduced by $A\alpha$ on each end. For the case of two solenoids each having more than one layer, formula (8) may be used, A and a being the mean radii, and n_1 and n_2 the total number of turns per centimeter in all the layers.

When the solenoids are very long in comparison with the radii, formula (8) may be simplified by omitting the terms $\frac{A^3}{r^3}$, $\frac{A^5}{r^5}$, etc. The expression for α then becomes

$$\alpha = \frac{1}{2} - \frac{a^2}{16A^2} - \frac{a^4}{128A^4} - \frac{5a^6}{2048A^6}. \tag{9}$$

Example. —

$$a = 5\text{ cm.},\quad A = 10\text{ cm.},\quad l = 200\text{ cm.}$$

Substituting these values in (8) for α we have

$$\alpha = 0.4875 - 0.0156 - 0.0005 + \cdots$$
$$= 0.4714,$$
$$M = 4\pi^2 \times 25\, n_1 n_2 (200 - 9.428) = 19057.2\,\pi^2 n_1 n_2.$$

If $n_1 = n_2 = 10$ turns per centimeter,

$M = 0.0188$ henry.

For two concentric coaxial solenoids of which the inner solenoid is considerably shorter than the other, we have the following formula:

$$M = 4\pi^2 a^2 n_1 n_2 \left[\rho_2 - \rho_1 + \frac{a^2A^2}{8}\left(\frac{1}{\rho_1{}^3} - \frac{1}{\rho_2{}^3}\right) - \frac{a^4A^2}{16}\left(\frac{1}{\rho_1{}^5} - \frac{1}{\rho_2{}^5}\right) + \frac{5\,a^4A^4}{64}\left(\frac{1}{\rho_1{}^7} - \frac{1}{\rho_2{}^7}\right) + \frac{5\,a^6A^2}{128}\left(\frac{1}{\rho_1{}^7} - \frac{1}{\rho_2{}^7}\right) - \cdots\right], \tag{10}$$

in which

$$\rho_1 = \sqrt{l_1{}^2 + A^2}, \quad \text{where } l_1 = \frac{x - l}{2},$$

$$\rho_2 = \sqrt{l_2{}^2 + A^2}, \quad \text{where } l_2 = \frac{x + l}{2}.$$

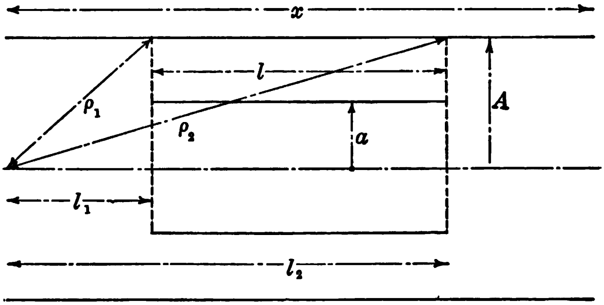

FIG. 14.

$l = l_2 - l_1$ = length of inner solenoid,
x = length of outer solenoid,
A and a are the radii of the solenoids.

This is a very accurate formula, and the series being rapidly convergent is very convenient for computation.

Example. —

$l = 5$ cm., $x = 30$ cm., $A = 5$ cm., $a = 4$ cm., $n_1 = 10$, $n_2 = 40$,
$\rho_1 = 13.463, \quad \rho_2 = 18.200,$
$M = 4\pi^2 \times 16 \times 400\{4.737 + 0.0122 - 0.0007\} = 11.99897$ cm.
= 0.0012 henry.

In the case of two coaxial coils or solenoids of the same diameter and the same number of turns per cm., we can also calculate the mutual inductance by the aid of self-inductance formulæ.

Let L_1, L_2, L_3 denote the self inductances of the three coils, 1, 2 and 3 respectively.

L = total inductance of the three coils in series,
L' = inductance of coils 1 and 2 in series,
L'' = inductance of coils 2 and 3 in series,
$L = L_1 + L_2 + L_3 + 2\,M_{12} + 2\,M_{23} + 2\,M_{13}$,
$L' = L_1 + L_2 + 2\,M_{12}$,
$L'' = L_2 + L_3 + 2\,M_{23}$,
$L - L' - L'' = - L_2 + 2\,M_{13}$,

and
$$M_{13} = \frac{L + L_2 - L' - L''}{2}. \tag{11}$$

The mutual inductance between coils 1 and 3 is expressed in terms of self inductance. For the evaluation of the self inductances in (11) see formulæ (13) and (14).

Self Inductance of a Single Layer Coil or Solenoid. — The following approximate formula for the self inductance of long solenoids is often given:

$$L = 4\,\pi^2 a^2 n_1^2 l, \tag{12}$$

where a is the mean radius, n_1 is the number of turns of wire per cm. and l is the length, supposed great in comparison with a. There is a considerable error in this formula due to the end effects.

Prof. Nagaoka* has recently published a very convenient and accurate formula for the calculation of the self inductance of single layer coils which is of the following form:

$$L = 4\,\pi^2 a^2 n_1^2 l K. \tag{13}$$

a, n_1 and l have the same significance as in formula (12) and K is a correction factor for the end effects, and is proportional to $\frac{2\,a}{l}$. Prof. Nagaoka has also prepared a table, which is reproduced here (Table V, p. 80), giving the values of K for values of $\frac{2a}{l}$ from 0.01 to 10. Formula (13) is very accurate and, with the aid of the table, very convenient for calculations.

For very short coils, we also have the following convenient formula:

$$L = 4\,\pi a n^2 \left\{ \log_e \frac{8\,a}{l} - \frac{1}{2} + \frac{l^2}{32\,a^2}\left(\log_e \frac{8\,a}{l} + \frac{1}{4} \right) \right\}. \tag{14}$$

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* *Jour. Coll. Sci.*, Tokyo, Art. 6, pp. 18–33; 1909.

n is the whole number of turns of wire on the coil, and a as before is the radius measured to the center of the wire.

Example. —

$$a = 10, \qquad l = 5 \text{ cm.}, \qquad n = 10, \frac{2a}{l} = 4.$$

From Table V we find $K = 0.3654$.

$$L = 4\pi^2 \times 100 \times 100 \times 5 \times 0.3654 = 721{,}300 \text{ cm.} = 0.7213 \text{ mh.}$$

Computing the same example by (14) we have

$$\log_e \frac{8a}{l} = \log_e 16 = 2.7725, \qquad \frac{b^2}{32a^2} = 0.0078, \quad n = 50,$$

$$L = 4\pi \times 10 \times 2500 \{2.7725 - 0.5 + 0.0078 (2.7725 + 0.25)\}$$
$$= 720{,}714 \text{ cm.} = 0.7207 \text{ mh.},$$

which agrees very closely with the result obtained by formula (13), formula (13) giving the more accurate result.

For shorter coils the two formulæ give results in still closer agreement.

In the derivation of formulæ (13) and (14) a uniform distribution of current over the entire surface of the cylinder was assumed, a condition which could be realized only if the winding were of infinitely thin strip and completely covered the solenoid. A winding of insulated wire or of bare wire in a screw thread may have a greater or less self inductance than that given above by the current sheet formulæ according to the ratio of the diameter of the wire to the pitch of the winding. Putting L for the actual value of the self inductance of a winding and L_s for the current sheet value given by (13) or (14), then

$$L = L_s - \Delta L.$$

The correction ΔL is given by the following expression:

$$\Delta L = 4\pi a n (A + B). \tag{15}$$

a is the radius, n is the whole number of turns and A and B are constants given in Tables VI and VII.

The correction term A depends on the diameter d of bare wire and the pitch, that is, on the value of the ratio $\frac{D}{d}$.

If in above example we assume $d = 0.08$ cm., then $\frac{d}{D} = \frac{0.08}{0.10} = 0.8$. From tables (VI, p. 82) and (VII, p. 83) we get

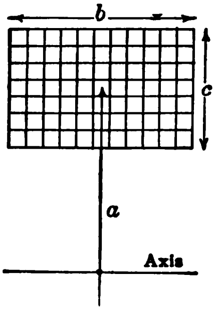

Fig. 15.

$$A = 0.3337, \qquad B = 0.3186,$$
$$\Delta L = 4\pi \times 10 \times 50\ (0.337 + 0.3186) = 4098,$$
$$L_s = 721,300,$$

and $L = 721,300 - 4098 = 0.7172$ mh.

The correction in this case is about 0.6 per cent.

The Self Inductance of a Circular Coil of Rectangular Section. — If the coil is not very long, the length being less than the diameter, we have the following expression for the self inductance of a circular coil of rectangular section, which is very accurate.

$$L = 4\pi a n^2 \left\{ \left(1 + \frac{3b^2 + c^2}{96\,a^2}\right) \log_e \frac{8a}{\sqrt{b^2 + c^2}} - y_1 + \frac{b^2}{16\,a^2} y_2 \right\}. \quad (16)$$

a = mean radius of coil, b and c breadth and depth of coil,

n = total number of turns.

The values of y_1 and y_2 are given in Table VIII, p. 83, as functions of $x = \frac{b}{c}$ or $\frac{c}{b}$, that is, x is the ratio of the breadth to depth of the section or vice versa, being always less than unity.

Example. —

$$a = 10 \text{ cm.}, \quad b = 1 \text{ cm.}, \quad c = 1 \text{ cm.}, \quad n = 100.$$

From Table VIII we find

$$y_1 = 0.8483, \quad y_2 = 0.8162,$$
$$\log_e \frac{8\,a}{\sqrt{b^2 + c^2}} = \log_e \frac{80}{\sqrt{2}} = 4.0354,$$
$$L = 4\pi \times 10 \times 10^4 \left\{ \left(1 + \tfrac{4}{9600}\right) 4.0354 - 0.8483 + 0.0005 \right\}$$
$$= 4\pi \times 318,900 = 4.0078 \text{ mh.}$$

For the self inductance of long solenoids of several layers, we have the following approximate formula:*

* Louis Cohen, *Electrical World,* Vol. 50, 1907.

$$L = 4\pi^2 n^2 m \left\{ \frac{2a_0^4 + a_0^2 l^2}{\sqrt{4a_0^2 + l^2}} - \frac{8a_0^3}{3\pi} \right\} + 8\pi^2 n^2 \left\{ [(m-1)\,a^2_1 \right.$$
$$+ (m-2)\,a_2^2 + \ \ldots\ ]\left(\sqrt{a_1^2 + l^2} - \frac{7}{8}a_1\right)$$
$$\left. + \frac{1}{2}[m\,(m-1)\,a_1^2 + \ \ldots\ ]\left(\frac{a_1\delta a}{\sqrt{a_1^2 + l^2}} - \delta a\right) \right\}, \qquad (17)$$

where a_0 is the mean radius of the solenoid, a_1, a_2, a_3 . . . a_m are the mean radii of the various layers, m is the number of layers, δa is the distance between centers of any two consecutive layers, and n is the number of turns per unit length.

For long solenoids, where the length is, say, four times the diameter, we can neglect the last term in equation (17).

This formula is sufficiently accurate for most purposes; it will give results accurate to within two per cent even for short solenoids, where the length is only twice the diameter.

Example. — A solenoid of length $l = 50$ cm., mean radius 5 cm. and depth of winding 0.4 cm., is wound with four layers of wire of 500 turns each.

The constants in formula (17) have the following numerical values:

$$a_0 = 5, \quad \delta a = 0.1, \quad l = 50, \quad m = 4, \quad n = 10,$$
$$a_1 = 4.75,\ a_2 = 4.85\ \ldots\ \sqrt{4a_0^2 + l^2} = 50.99,\quad \sqrt{a_1^2 + l^2} = 50.225.$$

Introducing these values in (17) we get

$$L = 16\pi^2 n^2 (1144.3 + 3360 - 10.84 - 1.04)$$
$$= 70.56 \text{ mh.}$$

Self Inductance of Circular Ring. — The self inductance of a ring of circular section is given by the following formula:

$$4\pi a \left\{ \log_e \frac{8a}{\rho} - 1.75 \right\}, \qquad (18)$$

where a is the radius of the circle and ρ is the radius of its cross section.

Empirical Formula. — Recently Profs. Brooks and Turner* have obtained an empirical formula for the self inductance of coils, which is applicable to a fair degree of approximation for the calculation of the inductance of any closely-wound cylindrical coil, long or short, thick or thin, from the long solenoid to the single turn of fine wire.

---

* Morgan Brooks and H. M. Turner, "Inductance of Coils," *University of Illinois Bulletin*, Vol. IX, No. 10.

The formula is as follows:

$$L = \frac{4\pi^2 a^2 N^2}{b + c + R} F'F''. \tag{19}$$

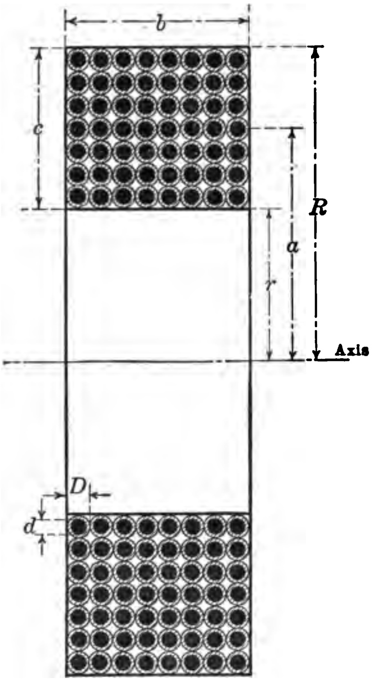

Fig. 16.

a = the mean radius of the winding,
b = the axial length of the coil; for single turns use diameter of wire,
c = the thickness of the winding,
R = the outer radius of the winding,
N = total number of turns in winding,

$$F' = \frac{10\,b + 12\,c + 2\,R}{10\,b + 10\,c + 1.4\,R},$$

$$F'' = 0.5 \log_{10}\left(100 + \frac{14\,R}{2\,b + 3\,c}\right).$$

For spaced windings, formula (19) is not recommended, especially when the space between the turns is great as compared with the diameter of the bare conductor.

For long coils, we may for a first approximation consider F' and F'' separately equal to unity, and formula (19) reduces to its first term

$$L = \frac{4\pi^2 a^2 N}{b + c + R}. \tag{20}$$

Formula (20) should be employed only when the length b is at least twice the outside diameter $2\,R$. The error involved scarcely exceeds 4 per cent with a multiple layer winding, and is within 2 per cent for a single layer solenoid, becoming more accurate as the length of the coil increases.

To check the accuracy of formula (19) we will work out by this formula the same examples as those given in illustration of formulæ (13), (16) and (17).

Example. —

$$a = 10 \text{ cm.}, \quad b = 5, \quad c = \text{diameter of wire} = 0.10,$$
$$N = 50, \quad R = 10.05,$$
$$F' = \frac{50 + 1.2 + 20.1}{50 + 1 + 14.07} = 1.096,$$
$$F'' = 0.5 \log_{10}\left(100 + \frac{140.7}{10.3}\right) = 1.0278,$$
$$L = \frac{4\pi^2 \times 100 \times 2500}{15.15} \times \frac{1.096 \times 1.0278}{10^6} = 0.734 \text{ mh.}$$

By formula (13) the value of the inductance for this coil is 0.7207 mh., hence the error in formula (19) for this case is about 2 per cent.

Example. —

$$a = 10, \quad b = 1, \quad c = 1, \quad N = 100, \quad R = 10.5,$$
$$F' = \frac{10 + 12 + 21}{10 + 10 + 14.7} = 1.239,$$
$$F'' = 0.5 \log_{10}\left(100 + \frac{14 \times 10.5}{5}\right) = 1.0566,$$
$$L = \frac{4\pi^2 \times 100 \times 10^4}{12.5} \times 1.239 \times 1.0566 = 4.13 \text{ mh.}$$

By formula (16) the value of the inductance for this coil is $L = 4.008$ mh., and the error in formula (19) for this case is 3 per cent.

Example. —

$$a = 5, \quad b = 50, \quad c = 0.4, \quad N = 2000, \quad R = 5.2,$$
$$F' = 1.007, \quad F'' = 1,$$
$$L = \frac{4\pi^2 \times 25 \times 4 \times 10^6}{50 + 0.4 + 5.2} \times \frac{1.007}{10^6} = 71.14 \text{ mh.}$$

By formula (17) the value of the inductance for this coil is

$$L = 70.56 \text{ mh.}$$

which agrees with the above within 0.8 per cent.

As an extreme case we may consider that of a single turn.

Let $a = 10$ cm., $\rho = 0.5$ cm.

$$\log_e \frac{8a}{\rho} = \log_e 160 = 5.075,$$

and by formula (18) we get

$$L = 4\pi 10 \{5.075 - 1.75\} = 417.83 \text{ cm.}$$

Working out the same example by formula (19) we find

$$L = 413 \text{ cm.},$$

the error being only 1 per cent.

From these examples it is evident that formula (19) can be relied upon to give results with an accuracy of from 1 to 3 per cent.

Table IX, p. 84, is reproduced from Prof. Brooks' and Turner's paper giving the values of F' and F'' for various coil proportions.

Self Inductance of Rectangle. —

Let a = length of rectangle,
b = breadth of rectangle,
ρ = radius of conductor,
$d = \sqrt{a^2 + b^2}$.

The formula for the self inductance is

$$L = 4 \left\{ (a + b) \log_e \frac{2\,ab}{\rho} - a \log_e (a + d) - b \log_e (b + d) - \frac{7}{4} (a + b) + 2 (d + \rho) \right\}. \quad (21)$$

For a rectangle made up of a conductor of rectangular section $\alpha \times \beta$,

$$L = 4 \left\{ (a + b) \log_e \frac{2\,ab}{\alpha + \beta} - a \log_e (a + d) - b \log_e (b + d) - \frac{a + b}{2} + 2\,d + 0.447\,(\alpha + \beta) \right\}. \quad (22)$$

For $a = b$, a square,

$$L = 8\,a \left\{ \log_e \frac{a}{\alpha + \beta} + 0.2235 \frac{\alpha + \beta}{a} + 0.726 \right). \quad (23)$$

Mutual Inductance of Two Equal Parallel Rectangles. — For two equal parallel rectangles of sides a and b and distance apart d the mutual inductance, which is the sum of the several mutual inductances of parallel sides, is

$$M = 4 \left[a \log_e \left(\frac{a + \sqrt{a^2 + d^2}}{a + \sqrt{a^2 + b^2 + d^2}} \cdot \frac{\sqrt{b^2 + d^2}}{d} \right) + b \log_e \left(\frac{b + \sqrt{b^2 + d^2}}{b + \sqrt{a^2 + b^2 + d^2}} \cdot \frac{\sqrt{a^2 + d^2}}{d} \right) \right] + 8 \left[\sqrt{a^2 + b^2 + d^2} - \sqrt{a^2 + d^2} - \sqrt{b^2 + d^2} + d \right]. \quad (24)$$

For a square, where $a = b$, we have

$$M = 8\left[a \log_e \left(\frac{a + \sqrt{a^2 + d^2}}{a + \sqrt{2\,a^2 + d^2}} \cdot \frac{\sqrt{a^2 + d^2}}{d} \right]\right. \\ + 8\,[\sqrt{2\,a^2 + d^2} - 2\sqrt{a^2 + d^2} + d]. \qquad (25)$$

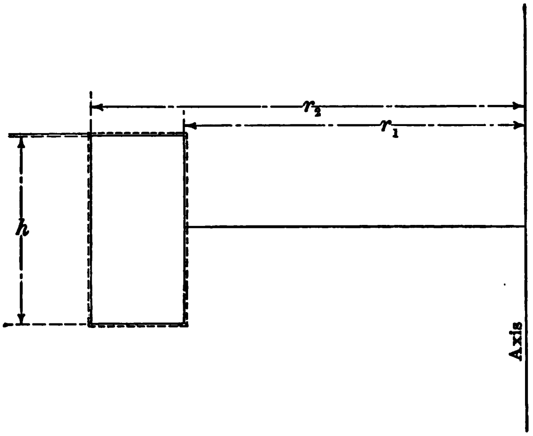

Fig. 17.

Self Inductance of Toroidal Coil. — When the section of the coil is rectangular, and it is wound uniformly with a single layer of n turns, the inductance is given by the following expression:

$$L = 2\,n^2 h \log_e \frac{r_2}{r_1}. \qquad (26)$$

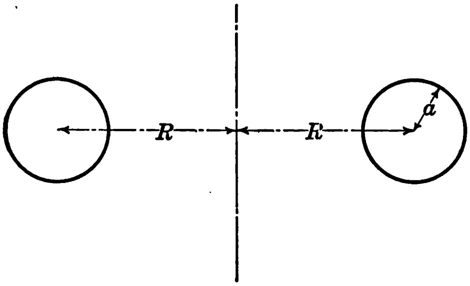

Fig. 18.

r_2 and r_1 are the outer and inner radii and h is the axial depth of the coil. For a coil of circular cross section we have for the inductance,*

$$L = 4\,\pi n^2 (R - \sqrt{R - a^2}), \qquad (27)$$

---

* "A Treatise on Alternating Current Theory," by Alex. Russell, Vol. I, p. 52.

where R is the mean radius of the coil and a is the radius of its cross section.

Inductance of Linear Conductors. — In problems dealing with the transmission of electrical energy by alternating currents over long distances, we must know accurately the electrical constants, inductance, capacity and resistance of the lines. We shall therefore try as far as possible to give here a complete list of all available formulæ for the inductance of linear conductors of various arrangement of circuits.

The simplest case is that of a single straight cylindrical wire, whose inductance is *

$$\left.\begin{aligned} L &= 2\left\{ l \log_e \frac{l+\sqrt{l^2+a^2}}{a} - \sqrt{l^2+a^2} + \frac{l}{4} + a \right\} \\ &= 2l\left\{ \log_e \frac{2l}{a} - \frac{3}{4} \right\}, \text{ approximately.} \end{aligned}\right\} \tag{28}$$

l = length of wire in cm. and a = radius of wire in cm. When the permeability of the wire is μ and that of the medium outside the wire is unity, formula (28) assumes the following form:

$$L = 2l\left\{ \log_e \frac{2l}{a} - 1 + \frac{\mu}{4} \right\}. \tag{29}$$

The mutual inductance between two parallel wires separated by a distance d is

$$\left.\begin{aligned} M &= 2\left\{ l \log_e \frac{l+\sqrt{l^2+d^2}}{d} - \sqrt{l^2+d^2} + d \right\} \\ &= 2l\left\{ \log_e \frac{2l}{d} + \frac{d}{l} - 1 \right\} \text{ approximately.} \end{aligned}\right\} \tag{30}$$

If in the case of two parallel wires one is the return circuit of the other, then the effective inductance of both wires is $L_1 + L_2 - 2M$ and the effective inductance of each wire separately is $\frac{L_1 + L_2 - 2M}{2}$. When the two wires are of the same diameter, the effective inductance of each wire is $L - M$. Using the values of L and M from (28) and (30) we get

$$L = 2l\left\{ \log_e \frac{d}{a} + \frac{1}{4} - \frac{d}{l} \right\}. \tag{31}$$

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* E. B. Rosa and Louis Cohen, *Bull. Bureau of Stand.*, Vol. 5, pp. 1–132.

Usually $\frac{d}{l}$ is very small, hence

$$L = 2l\left\{\log_e \frac{d}{a} + \frac{1}{4}\right\}. \tag{32}$$

If the wires are of magnetic permeability μ,

$$L = 2l\left\{\log_e \frac{d}{a} + \frac{\mu}{4}\right\}. \tag{33}$$

The total inductance of the circuit including the to and fro conductors is

$$L = 4l\left\{\log_e \frac{d}{a} + \frac{1}{4}\right\}. \tag{34}$$

Equations (28) to (34) give the inductances in centimeters. To express the inductance in mh. per km., we must multiply each of the formulæ by the numerical factor 0.1. To express the inductances in mh. per mile, we must multiply each of the formulæ by the numerical factor 0.161.

If we replace the Naperian logarithms by the common logarithms, we may put (34) in the following form:

$$\left.\begin{aligned} L &= 1.483 \log_{10} \frac{d}{a} + 0.161 \text{ mh. per mile} \\ &= 0.921 \log_{10} \frac{d}{a} + 0.10 \text{ mh. per km.,} \end{aligned}\right\} \tag{35}$$

d and a being expressed in the same units, inches or centimeters.

Formulæ (28) to (35) were derived on the assumption of uniform current distribution over the entire sectional areas of conductors, which is only strictly true for continuous currents. For alternating currents formula (34) assumes the form

$$L = 4l\left\{\log_e \frac{d}{a} + \alpha\right\}, \tag{36}$$

where α is a constant the value of which depends on the frequency. For low frequencies we can take α equal to $\frac{1}{4}$ and for very high frequencies α is zero. An accurate expression for α has been given by Professor Pedersen,* which is as follows:

$$\alpha = \frac{\mu}{2 \times 10^9} l_1 (ma), \tag{37}$$

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* P. O. Pedersen, *Jahrbuch der Drahtlosen Teleg.*, Vol. 4, pp. 501–515.

where n = frequency in cycles per second,

$m = 2\pi\sqrt{\frac{2n\mu}{\rho}}$,

μ = permeability,

ρ = resistivity in C.G.S. units,

a = radius of wire in cm.

The values of $l_1\,(ma)$ for values of ma from 0 to 6 are given in Table II, p. 43, Chap. I. For values of ma greater than 6 we have the following approximate expression:

$$l_1\,(ma) = \frac{1.4142}{ma} - \frac{0.5303}{(ma)^3} - \frac{0.75}{(ma)^4}. \tag{38}$$

Example. — What is the value of α for a copper conductor 0.1 cm. radius and frequency 20,000?

$$m = 2\pi\sqrt{\frac{4 \times 10^4 \times 5.9}{10^4}} = 30.52,$$

$$ma = 3.05.$$

From Table II we find $l_1\,(ma) = 0.4188$,
hence $\alpha = 0.2094$.

In Table X, p. 85, we give the values of L calculated by formula (35) for various sizes of B. & S. wire, and different distances apart.

Inductance of Three-phase Transmission Lines. — The inductance of one wire of a three-phase circuit arranged on the corners of an equilateral triangle, is the same as in the case of a two-wire circuit with the wires the same distance apart.

$$L_1 = L_2 = L_3 = 2\,l\left\{\log_e \frac{d}{a} + \frac{1}{4}\right\}, \tag{39}$$

where d = distance between any two wires and a = radius of wire.

To get the inductive drop between two wires, the inductances are combined geometrically, the inductance of circuits (1.2), for example, being

$$L_{12} = 2\sqrt{3}\,l\left\{\log_e \frac{d}{a} + \frac{1}{4}\right\}. \tag{40}$$

If the wires forming the circuit are arranged in a straight line we have for the inductance of the middle wire

$$L_1 = 2\,l\left\{\log_e \frac{d}{a} + \frac{1}{4}\right\},$$

while the inductance of the outer conductors, assuming equal load distribution on all three phases, is as follows:

$$L_2 = L_3 = 2\,l\left\{\log_e \frac{d}{a} + 0.625\right\}. \tag{41}$$

To equalize the inductances of all three phases the wires are transposed, each wire taking the center pin for one-third the distance. Arranged in this way the inductance of each wire is

$$L = 2l\left\{\log_e \frac{d}{a} + 0.48\right\}, \tag{42}$$

$$L_{12} = 2\sqrt{3}\,l\left\{\log_e \frac{d}{a} + 0.48\right\}.$$

In equations (39) to (42) the length of the lines and the inductances are expressed in centimeters. Expressing the inductances in mh., km. or mile, and at the same time convert the Naperian logarithms into common logarithms, equations (40) and (42) assume the following form:

$$\left.\begin{aligned} L_{12} &= 0.797 \log_{10} \frac{d}{a} + 0.0866 \text{ mh. per km.} \\ &= 1.283 \log_{10} \frac{d}{a} + 0.14 \text{ mh. per mile.} \end{aligned}\right\} \tag{40'}$$

$$\left.\begin{aligned} L_{12} &= 0.797 \log_{10} \frac{d}{a} + 0.166 \text{ mh. per km.} \\ &= 1.283 \log_{10} \frac{d}{a} + 0.27 \text{ mh. per mile.} \end{aligned}\right\} \tag{42'}$$

Mutual Inductance.* — Two metallic circuits A, B and C, D suspended on the same pole and arranged in the form shown in Fig. (19). The mutual inductance between A and CD is

$$M_{A-CD} = 2\,l \log_e \frac{d_1}{d_2};$$

the mutual inductance between B and CD is

$$M_{B-CD} = 2\,l \log_e \frac{d_3}{d_4}.$$

Since the current in A is equal to that in B but of opposite sign, we have for the mutual inductance between the two circuits

$$M_{AB-CD} = 2\,l\left\{\log_e \frac{d_1}{d_2} - \frac{d_3}{d^4}\right\}. \tag{43}$$

---

* "Theorie der Wechselströme," von J. L. La Cour and O. S. Bragstad, pp. 547–555.

If the circuit CD consists of only one conductor C, the earth being the return conductor, then we may put to a high degree of approximation $d_1 = d_3$

and
$$M_{AB-C} = 2\,l \log_e \frac{d_4}{d_2}. \tag{44}$$

In case it is desired to avoid inductive effects the conductors must be so arranged as to make $d_1d_4 = d_2d_3$, so that

$$\log_e \frac{d_1d_4}{d_2d_3} = \log_e 1 = 0.$$

Such an arrangement of a pair of circuits is shown in Fig. 20.

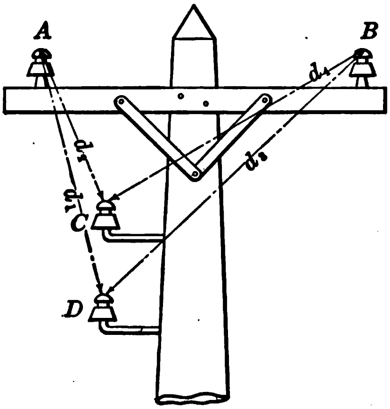

Fig. 19.

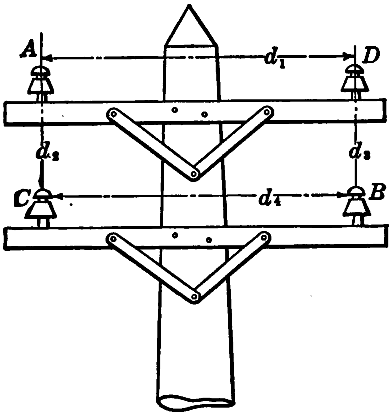

Fig. 20.

Grounded Conductors.* — The self inductance of a linear conductor suspended at height h above ground, and the ground being the return conductor, is

$$\left.\begin{aligned} L &= 2\,l\left\{\log_e \frac{2\,h}{a} + \frac{1}{4}\right\} \\ &= 0.74 \log_{10} \frac{d}{a} + 0.08 \text{ mh. per mile;} \end{aligned}\right\} \tag{45}$$

a is the radius and l is the length of conductor.

The mutual inductance between two grounded parallel wires is

$$\left.\begin{aligned} M &= l\left\{\log_e \frac{d^2 + (h_1 + h_2)^2}{d^2 + (h_1 - h_2)^2}\right. \\ &= 0.37 \log_{10} \frac{d^2 + (h_1 + h_2)^2}{d^2 + (h_1 - h_2)^2} \text{mh. per mile,} \end{aligned}\right\} \tag{46}$$

---

* O. Heaviside, *Collected Papers*, Vol. 1, p. 101.

where h_1 and h_2 are the heights above ground of wires 1 and 2 respectively and d is the distance between them.

When the wires are at the same height above ground, equation (45) reduces to

$$M = 0.74 \log_{10} \sqrt{\frac{4h^2 + d^2}{d}} \text{ mh. per mile.} \qquad (47)$$

Inductance of Split Conductors.[*] — The use of split conductors has been proposed for long distance transmission lines as a means of reducing the inductance and increasing the capacity of the line.

Consider first the case of a looped conductor split into two parts, the two parts of each conductor being placed symmetrically with respect to each other, and the distance between the centers of gravity, D, as shown in Fig. 21. Let a denote the radius of an equivalent conductor whose sectional area is equal to the sum of the sectional areas of the two parts of the split conductor. Then we have for the total inductance of the circuit including the to and fro conductor

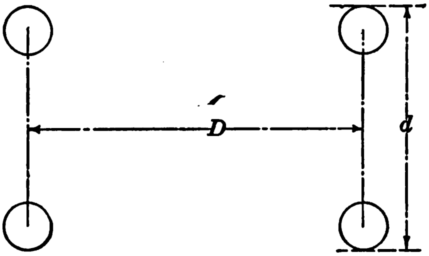

FIG. 21.

$$L = 2l \left\{ \log_e \frac{D}{a} + \log_e \frac{D}{d} + 0.597 \right\} \text{ cm.} \qquad (48)$$

or

$$L = 0.322 \left\{ \log_e \frac{D}{a} + \log_e \frac{D}{d} + 0.597 \right\} \text{ mh. per mile.} \qquad (49)$$

Example. —

Let $a = 0.1825$ in., $D = 48$ in., $d = 5a$,

$$\log_e \frac{D}{a} = 5.571, \quad \log_e \frac{D}{d} = 3.962,$$

$$L = 3.26 \text{ mh. per mile.}$$

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[*] Louis Cohen, "Inductance of Split Aerial Conductors," *Electrical World*, Nov., 1912.

For a solid conductor of the same radius and the same distance between outgoing and return conductor we find, by formula (34),

$$L = 3.84 \text{ mh. per mile.}$$

Hence, splitting the conductor into two parts and separating these parts by a distance equal to five times the equivalent radius of solid conductor reduces the inductance by about 15 per cent.

Conductor Split into Three Parts.* — Let the three parts of each conductor be arranged symmetrically at the vertices of an equilateral triangle as shown in Fig. 22. The inductance of the circuit including the to and fro conductor is

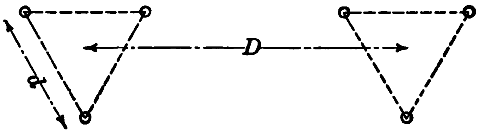

FIG. 22.

$$\left.\begin{aligned} L &= l\left\{\frac{2}{3}\log_e\frac{D}{a} + \frac{4}{3}\log_e\frac{D}{d} + 0.533\right\} \text{ cm.} \\ &= 0.322\left\{\frac{2}{3}\log_e\frac{D}{a} + \frac{4}{3}\log_e\frac{D}{d} + 0.533\right\} \text{mh. per mile.}\end{aligned}\right\} \quad (50)$$

Considering the same example we have worked out for the case of a conductor split into two parts, we have

$$D = 48 \text{ in.}, \quad a = 0.1825 \text{ in.}, \quad d = 5\,a$$

and $$L = 3.07 \text{ mh. per mile.}$$

By splitting the conductor into three parts and separating the parts by a distance equal to five times the radius, the inductance of the circuit is reduced by about 20 per cent.

Self Inductance of Two Parallel Hollow Cylindrical Conductors.† — Let a_2, a_1, and b_2, b_1 denote the outer and inner radii of the two cylinders respectively; d is the interaxial distance between the two conductors, and l the length of each conductor in centimeters. The total inductance of the circuit is

$$L = l\left\{2\log_e\frac{d^2}{a_2b_2} + \frac{2\,a_1^4}{(a_2^2 - a_1^2)}\log_e\frac{a_2}{a_1} + \frac{1}{2}\,\frac{a_2^2 - 3\,a_1^2}{a_2^2 - a_1^2}\right.$$
$$\left. + \frac{2\,b_1^4}{(b_2^2 - b_1^2)^2}\log_e\frac{b_2}{b_1} + \frac{1}{2}\,\frac{b_2^2 - 3\,b_1^2}{b_2^2 - b_1^2}\right\} \text{cm.} \quad (51)$$

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* l. c.

† Alex. Russell, "Theory of Alternating Currents," Vol. I, pp. 55–59.

If we put

$$h^2 = a_2^2 - a_1^2, \quad k^2 = b_2^2 - b_1^2,$$

equation (51) can be written in the following form:

$$L = l\left\{2\log_e\frac{d^2}{a_2b_2} + \frac{1}{3}\left(\frac{h^2}{a_2^2} + \frac{k^2}{b_2^2}\right) + \frac{1}{12}\left(\frac{h^4}{a_4^4} + \frac{k^4}{b_2^4}\right) + \cdots \right.$$

$$\left. \cdots + \frac{2}{n(n+1)(n+2)}\left(\frac{h^{2n}}{a_2^{2n}} + \frac{k^{2n}}{b_2^{2n}}\right)\right\}\text{ cm.} \qquad (52)$$

To reduce (50) and (51) so as to give the inductance in mh. per mile it is necessary to multiply the right-hand side of either equation by the numerical factor 0.161.

Self Inductance of Concentric Conductors. — The following formulæ apply to two hollow cylinders, one inclosed in the other, and insulated from it, the current entering one cylinder and returning by the other. Let the outer and inner radii of the outer and inner cylinders be denoted by b_2, b_1, a_2, a_1 and let l be the length in cms. The inductance is

$$L = l\left\{2\log_e\frac{b_1}{a_2} + \frac{2a_1^4}{(a_2^2 - a_1^2)^2}\log_e\frac{a_2}{a_1} + \frac{1}{2}\,\frac{a_2^2 - 3a_1^2}{a_2^2 - a_1^2}\right.$$

$$\left. + \frac{2b_2^4}{(b_2^2 - b_1^2)^2}\log_e\frac{b_2}{b_1} - \frac{1}{2}\,\frac{3b_2^2 - b_1^2}{b_2^2 - b_1^2}\right\}\text{ cm.} \qquad (53)$$

In practice the cross-sectional areas of the two cylinders are always equal, that is,

$$\pi(a_2^2 - a_1^2) = \pi(b_2^2 - b_1^2)$$

and formula (53) becomes

$$L = l\left\{2\log_e\frac{b_1}{a_2} + \frac{2}{(a_2^2 - a_1^2)^2}\left[a_1^4\log_e\frac{a_2}{a_1} + b_2^4\log_e\frac{b_2}{b_1}\right.\right.$$

$$\left.\left. - \frac{1}{2}(b_2^2 + a_1^2)(a_2^2 - a_1^2)\right]\right\}\text{ cm.} \qquad (54)$$

If the inner cylinder were solid, a_1 would be zero and b_2^2 would be equal to $b_1^2 + a_1^2$, and formula (54) would reduce to

$$\left.\begin{aligned} L &= l\left\{2\log_e\frac{b_1}{a_2} + \left(1 + \frac{b_1^2}{a_2^2}\right)^2\log_e\left(1 + \frac{a_2^2}{b_1^2}\right) - \frac{b_1^2}{a_2^2} - 1\right\} \\ &= l\left\{2\log_e\frac{b_1}{a_2} + \frac{1}{2} + \frac{1}{3}\,\frac{a_2^2}{b_1^2} - \frac{1}{12}\,\frac{a_2^4}{b_1^4} + \frac{1}{30}\,\frac{a_2^6}{b_1^6}\cdots\right\}\text{ cm.}\end{aligned}\right\} \quad (55)$$

The least possible value of L is when $b_1 = a_2$ and in this case

$$L = l\,(4\log_e 2 - 2) = 0.7726\,l.$$

For very high frequencies where the current is concentrated along the surface of the conductor, the magnetic force is confined within the space between the conductors, and in that case we have for the inductance

$$L = 2\,l \log_e \frac{b}{a}. \tag{56}$$

Equations (55) and (56) may be combined into one by writing them in the following form:

$$L = 0.161\,l \left\{ 2 \log_e \frac{b_1}{a_2} + \left[\frac{1}{2} + \frac{1}{3}\frac{a_2^2}{b_1^2} - \frac{1}{12}\frac{a_2^4}{b_1^4} + \cdots \right] \alpha \right\}, \tag{57}$$

where α is a quantity that depends on the frequency. For low frequencies it may be taken equal to unity, and for very high frequencies it is zero.

Example. —

$$b_1 = 0.4 \text{ in.}, \qquad a_2 = 0.2 \text{ in.},$$

$$\log_e \frac{b_1}{a_2} = 0.693,$$

$$\frac{1}{2} + \frac{1}{3}\frac{a_2^2}{b_1^2} - \frac{1}{12}\frac{a_2^4}{b_1^4} + \cdots = 0.578.$$

For low frequencies $\alpha = 1$ and

$$L = 0.316 \text{ mh. per mile.}$$

For very high frequencies $\alpha = 0$ and

$$L = 0.223 \text{ mh. per mile.}$$

Triple Concentric Conductors.* — The inductance of a triple concentric conductor depends on the current distribution in the three conductors. Consider three hollow cylinders of negligible thickness and radii r_1, r_2, r_3, beginning with the inner one. Let the current I_1 flow through the inner conductor and return currents I_2 and I_3 through the middle and outer conductors. Then

$$I_1 = I_2 + I_3.$$

The inductance of the system is

$$\left.\begin{aligned} L &= 2\,l \log_e \frac{r_2}{r_1} + 2\,l \left(\frac{I_3}{I_1}\right)^2 \log_e \frac{r_3}{r_2} \text{ cm.} \\ &= 0.322 \left\{ \log_e \frac{r_2}{r_1} + \left(\frac{I_3}{I_1}\right)^2 \log_e \frac{r_3}{r_2} \right\} \text{ mh. per mile.} \end{aligned}\right\} \tag{58}$$

---

* "Theory of Alternating Currents," by Alex. Russell, Vol. I, Chap. 2. See also "The Magnetic Circuit," by V. Karapetoff, Chap. XI.

If a current I_2 flows through the middle conductor and I_1, I_3 are the return currents through the inner and outer conductors,

$$L = 0.322 \left\{ \left(\frac{I_1}{I_2}\right)^2 \log_e \frac{r_2}{r_1} + \left(\frac{I_3}{I_2}\right)^2 \log_e \frac{r_3}{r_2} \right\} \text{mh. per mile.} \tag{59}$$

If a current I_3 flows through the outer conductor and the return currents I_1, I_2 through the inner and middle conductors,

$$L = 0.322 \left\{ \left(\frac{I_1}{I_3}\right)^2 \log_e \frac{r_2}{r_1} + \log_e \frac{r_3}{r_2} \right\} \text{mh. per mile.} \tag{60}$$

The Self Inductance of a Circuit Formed by Three Conductors Whose Axes Lie Along the Edges of an Equilateral Prism.— We will assume that the three conductors have equal radii a, the interaxial distance between any two conductors being d. Let a current I_1 flow through conductor No. 1 and the return currents I_2 and I_3 through conductors No. 2 and 3 respectively.

The inductance of the system is

$$\left.\begin{aligned} L &= l \left\{ 4 \log_e \frac{d}{a} + 1 \right\} \left\{ 1 - \frac{I_2 I_3}{I_1^2} \right\} \text{cm.} \\ &= 0.161 \left\{ 4 \log_e \frac{d}{a} + 1 \right\} \left\{ 1 - \frac{I_2 I_3}{I_1^2} \right\} \text{mh. per mile.} \end{aligned}\right\} \tag{61}$$

The inductance has a maximum value

$$L = 0.161 \left(4 \log_e \frac{d}{a} + 1 \right)$$

when either I_2 or I_3 is zero and it has a minimum value

$$L = 0.161 \left(4 \log_e \frac{d}{a} + 1 \right) \frac{3}{4}$$

when $I_2 = I_3 = \frac{1}{2} I_1$.

Compound Conductors. — A compound conductor consisting of a steel core and a concentric coating of copper is now being used in many cases for transmission purposes.

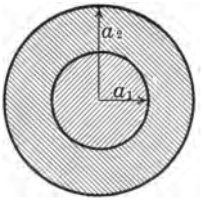

FIG. 23.

The inductance of a linear conductor consists of two parts, one of which is due to the magnetic field outside of the conductor and the other is due to the magnetic field within the conductor. We may call these the external and internal inductances, respectively. In the expression for the inductance of a linear conductor

$$L = l\left\{2\log_e\frac{d}{a} + \frac{\mu}{2}\right\}.$$

The first term is the external inductance, and the second term the internal inductance.

In compounding the conductor only the internal inductance is affected and the term $\frac{\mu}{2}$ is replaced by the following formula:*

$$L_i = \frac{\frac{1}{2}\mu_1 I_1^2 + 2\,\mu_2 I_1 I_2 + \frac{1}{2}\mu_2 I_2^2\left(1 - \frac{2\,a_1^2}{a_2^2 - a_1^2}\right) + 2\,\mu_2\left(I_1 - \frac{a_2^2 I^2}{a_2^2 - a_1^2}\right)\log_e\frac{a_2}{a_1}}{(I_1 + I_2)^2} \tag{62}$$

where a_1 = radius of core,
a_2 = radius of shell,
I_1 = current in core,
I_2 = current in shell,
μ_1 and μ_2 are the permeabilities of core and shell, respectively.

The above formula is somewhat complicated for numerical computation, and Mr. Fowle has supplied a short table in his paper, which we reproduce here, giving the values of L_i for the standard sizes now in commercial use.

TABLE (A)

INTERNAL INDUCTANCE OF COPPER-CLAD STEEL WIRES WHEN THE CONDUCTANCE RATIO OF STEEL TO COPPER IS 12 PER CENT

Values of n.	Internal inductance in C. G. S. units.	
Per cent		
20	$L_i = 0.149\,\mu_1$	$+ 0.0646\,\mu_2$
30	$L_i = 0.0506\,\mu_1$	$+ 0.102\,\mu_2$
40	$L_i = 0.0208\,\mu_1$	$+ 0.147\,\mu_2$
50	$L_i = 0.00925\,\mu_1$	$+ 0.193\,\mu_2$

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* The above formula was first given by Prof. A. Vaschy in his book, "Traite d'electricite et de Magnetisme," Vol. 1, p. 367 and was independently derived by Mr. F. F. Fowle, *Electrical World*, Dec. 29, 1910.

TABLE (B)

INTERNAL INDUCTANCE OF ALUMINUM-STEEL WIRES, WHEN THE CONDUCTANCE RATIO OF STEEL TO ALUMINUM IS 19.35 PER CENT

Values of n.	Internal inductance in C. G. S. units.	
Per cent		
20	$L_i = 0.127\,\mu_1$	$+\,0.0990\,\mu_2$
30	$L_i = 0.0328\,\mu_1$	$+\,0.1861\,\mu_2$
40	$L_i = 0.00871\,\mu_1$	$+\,0.2624\,\mu_2$

In the above two tables n is the conductance ratio of the compound wire to a solid copper wire of equal size.

Series or Parallel Arrangement of Inductance. — If several inductances are connected in series the total inductance is the sum of the several inductances.

$$L_t = L_1 + L_2 + L_3 + \cdots + L_n. \qquad (63)$$

For several inductances in parallel, we have for the total inductance,

$$L_t = \frac{L_1L_2L_3L_4 \cdots L_n}{L_2L_3L_4 \cdots L_n + L_1L_3L_4 \cdots L_n + L_1L_2L_4 \cdots L_n + \cdots}. \qquad (64)$$

If the mutual inductance between the various inductance coils is taken into account, then for two coils in series we have for the effective inductance,

$$L_t = L_1 + L_2 + 2\,M, \qquad (64)$$

or

$$L_t = L_1 + L_2 - 2\,M.$$

The first of the above two formulæ applies when the magnetic fields of the two coils are aiding, and the second formula applies when the magnetic fields are opposing.

For two coils in parallel we have

$$L_t = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2\,M}. \qquad (65)$$

For more than two inductances in parallel the formula for the total inductance is generally very complicated if the mutual inductance between the various branches is to be included. For the case, however, of cylindrical conductors symmetrically arranged, formulæ have been derived for the total inductance of several wires including the mutual inductance between the wires.*

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* "Inductance and Capacity of Linear Conductors," *The Electrician*, Feb. 14 and 21, 1913.

Such an arrangement of circuits is being used for instance in the case of horizontal antennæ which generally consist of a number of wires in parallel suspended at some height above ground.

Inductance of Wires in Parallel. — The following formulæ (66) to (72) were derived on the assumption that all the wires are suspended at the same height above ground, the earth being the return conductor; also that all the wires are of the same size and equally spaced.

For two wires in parallel

$$\mathcal{L} = \frac{L + M}{2}, \tag{66}$$

where $\mathcal{L}$ is the total inductance, L is the inductance of each individual wire and M is the mutual inductance between the wires. The values of L and M can be obtained by formulæ (45) and (48).

Example. —

a, radius of wire = 0.08 in.,
d, distance between wires = 24 in.,
h, height above ground = 80 ft.

By (45) and (48) we have

$$L = 2\left(\log_e \frac{2h}{a} + \frac{1}{4}\right) = 2 \log_e \left(\frac{160 \times 12}{0.08} + 0.25\right)$$

$$= 19.15 \text{ cm. per cm.}$$

$$M = \log_e \frac{4h^2}{d^2} = \log_e \frac{4 \times 6400}{2} = 8.76 \text{ cm. per cm.}$$

In the expression for M we neglected $\frac{d^2}{h^2}$ as being very small compared with unity.

We have by (66)

$$\mathcal{L} = \frac{19.15 + 8.76}{2} = 13.15 \text{ cm. per cm.}$$

For three wires in parallel

$$\mathcal{L} = \frac{L(L + M_2) - 2M_1^2}{3L + M_2 - 4M_1}. \tag{67}$$

M_1 is the mutual inductance between wires 1, 2 and 2, 3; M_2 is the mutual inductance between 1 and 3.

Since the wires are equally spaced, we may put $M_2 = M_1 - \log_e 4 = M_1 - 1.386$ and formula (67) becomes

$$\mathfrak{L} = \frac{L\,(L + M_1 - 1.386) - 2\,M_1^2}{3\,L - 3\,M_1 - 1.386}. \tag{68}$$

Using the same values for L and M as in the previous example we find the value of the inductance for three wires.

$$\mathfrak{L} = \frac{19.15\,(19.15 + 8.76 - 1.386) - 2\,(8.76)^2}{3 \times 19.15 - 3 \times 8.76 - 1.386} = 11.9 \text{ cm. per cm.}$$

For four wires in parallel

$$\mathfrak{L} = \frac{(L + M_1)\,(L + M_3) - (M_1 + M_2)^2}{4\,L - 4\,M_2 - 2\,M_1 + 2\,M_3}, \tag{69}$$

where
$$M_1 = M_{12} = M_{23} = M_{34},$$
$$M_2 = M_{13} = M_{24},$$
$$M_3 = M_{14}.$$

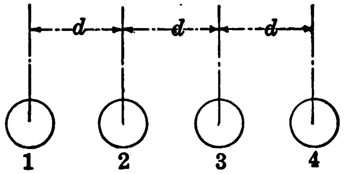

FIG. 24.

Formula (69) may be put in the form

$$\mathfrak{L} = \frac{(L + M_1)\,(L + M_1 - 2.2) - (2\,M_1 - 1.386)^2}{4\,L - 4\,M_1 + 1.15.}. \tag{70}$$

Taking again the values of $L = 19.15$ and $M_1 = 8.76$, we get for the inductance of four wires in parallel,

$$L = 10.93 \text{ cm. per cm.}$$

For five wires in parallel,

$$\mathfrak{L} = \frac{L\,(L^2+LM_2+LM_4+M_2M_4-2M_2^2-3M_1^2-2M_1M_3-M_3^2)+M_1\,(4M_1M_2+4M_2M_3-2M_1M_4)-2M_2^2}{L\,(5L-8M_1-4M_3+3M_4-M_2)+M_1\,(2M_3+8M_2-4M_4-M_1)+M_2\,(4M_1+4M_3-8M_2+M_4)-M_3^2} \tag{71}$$

where
$$M_1 = M_{12} = M_{23} = M_{34} = M_{45},$$
$$M_2 = M_{13} = M_{24} = M_{35},$$
$$M_3 = M_{13} = M_{25},$$
$$M_4 = M_{15}.$$

Expressing the inductance in terms of L and M_1 only, as in the previous cases, formula (71) reduces to

$$\mathfrak{L} = \frac{L^3 + L^2(2M_1 - 4.16) + L(-7M_1^2 + 10.24\,M_1 - 4.88) + M_1(6M_1^2 - 14.4\,M_1 + 12.2) - 2(M_1 - 1.4)^3}{5L^2 - L(10M_1 - 1.86) + 5M_1^2 - 1.87\,M_1 - 4.18}. \tag{72}$$

For the values $L = 19.15$ and $M_1 = 8.76$

$$\mathfrak{L} = \frac{7022.7 + 4899.4 - 8662.3 + 3038.8 - 949}{1833.6 - 1642 + 383.7 - 16.38 - 4.18}$$

$$= \frac{5349.5}{554.8} = 9.64 \text{ cm. per cm.}$$

If we have more than five wires in parallel, which is generally the case for antennæ of large stations, we can obtain an approximate value of the inductance in the following manner: Suppose we have an antenna consisting of ten wires, determine the inductance of five wires in parallel and denote it by L', then assume that the antenna consists of two wires separated by a distance $5\,d$ (where d is the distance between adjacent wires) and the inductance of each wire is L'. By formula (66), the effective inductance of the ten wires in parallel is

$$\mathfrak{L} = \frac{L' + M'}{2},$$

where M' is the mutual inductance between two wires separated by a distance $5\,d$.

Taking the data from the previous examples, we have for the inductance of five wires in parallel

$$L' = 9.64 \text{ cm.}$$

By (48) we have

$$M' = \log_\epsilon \frac{4\,(80)^2}{10^2} = 5.54 \text{ cm.};$$

hence $$\mathfrak{L} = \frac{9.64 + 5.54}{2} = 7.6 \text{ cm. per cm.}$$

If the antenna were 200 feet long, we should have

$$\mathfrak{L} = 7.6 \times 200 \times 30.5 = 0.046 \text{ mh.}$$

If the antenna consisted of twelve wires, for instance, we could break it up into three groups of four wires each and proceed in the same manner as above using formulæ (68) and (70).

The above formulæ for the inductance of parallel wires are very useful in the determination of the capacity of antennæ, as will be shown in Chapter III.

TABLE IV

(For use in Formula 4.)

q.	$\log_{10}\left[\frac{-\sqrt{Aa}}{z}\frac{\partial M}{\partial z}\right]$.	Δ.
0.020	$\bar{1}$.03365	10421
0.022	$\bar{1}$.13786	9528
0.024	$\bar{1}$.23314	8775
0.026	$\bar{1}$.32091	8140
0.028	$\bar{1}$.40231	7592
0.030	$\bar{1}$.47823	7115
0.032	$\bar{1}$.54938	6697
0.034	$\bar{1}$.61635	6328
0.036	$\bar{1}$.67963	5999
0.038	$\bar{1}$.73962	5706
0.040	$\bar{1}$.79668	5440
0.042	$\bar{1}$.85108	5201
0.044	$\bar{1}$.90309	4983
0.046	$\bar{1}$.95292	4785
0.048	0.00077	4604
0.050	0.04681	4437
0.052	0.09118	4283
0.054	0.13401	4141
0.056	0.17542	4009
0.058	0.21551	3888
0.060	0.25439	3774
0.062	0.29213	3668
0.064	0.32881	3569
0.066	0.36450	3477
0.068	0.39927	3390
0.070	0.43317	3308
0.072	0.46625	3232
0.074	0.49857	3160
0.076	0.53017	3091
0.078	0.56108	3028
0.080	0.59136	2967
0.082	0.62103	2910
0.084	0.65013	2855
0.086	0.67868	2804
0.088	0.70672	2754
0.090	0.73428	2709
0.092	0.76137	2665
0.094	0.78802	2623
0.096	0.81425	2584
0.098	0.84009	2546
0.100	0.86555	1259
0.101	0.87814	1251
0.102	0.89065	1242
0.103	0.90307	1233
0.104	0.91540	1226
0.105	0.92766	1218
0.106	0.93984	1209
0.107	0.95193	1202
0.108	0.96395	1195
0.109	0.97590	1188
0.110	0.98778	1180
0.111	0.99958	1174
0.112	1.01132	1166
0.113	1.02298	1161
0.114	1.03459	1153
0.115	1.04612	1148
0.116	1.05760	1141
0.117	1.06901	1135
0.118	1.08036	1130
0.119	1.09166	1123
0.120	1.10289	1118
0.121	1.11407	1113
0.122	1.12520	1107
0.123	1.13727	1102
0.124	1.14729	1097
0.125	1.15826	1091
0.126	1.16917	1087
0.127	1.18004	1082
0.128	1.19086	1078
0.129	1.20164	1073
0.130	1.21237	1068
0.131	1.22305	1064
0.132	1.23369	1060
0.133	1.24429	1055
0.134	1.25484	1052
0.135	1.26536	1048
0.136	1.27584	1043
0.137	1.28627	1040
0.138	1.29667	1036
0.139	1.30703	1033
0.140	1.31736	1029
0.141	1.32765	1025
0.142	1.33790	1022
0.143	1.34812	1019
0.144	1.35831	1016
0.145	1.36847	1012
0.146	1.37859	1010
0.147	1.38869	1006
0.148	1.39875	1004
0.149	1.40879	1000
0.150	1.41879	

TABLE V

Nagaoka's Table of Values of the End Correction K as Function of the Ratio $\frac{\text{Diameter}}{\text{Length}}$.

(For use in Formula 13.)

$\frac{\text{Diameter}}{\text{Length}}$.	K.	Δ_1.	Δ_2.	$\frac{\text{Diameter}}{\text{Length}}$.	K.	Δ_1.	Δ_2.
0.00	1.000000	−4231	+24	0.45	0.833723	−3160	+21
0.01	1.995769	−4207	26	0.46	0.830563	−3139	22
0.02	1.991562	−4181	24	0.47	0.827424	−3117	21
0.03	1.987381	−4157	25	0.48	0.824307	−3096	21
0.04	1.983224	−4132	25	0.49	0.821211	−3075	21
0.05	0.979092	−4107	+25	0.50	0.818136	−3054	+21
0.06	0.974985	−4082	26	0.51	0.815082	−3033	21
0.07	0.970903	−4056	24	0.52	0.812049	−3012	21
0.08	0.966847	−4032	24	0.53	0.809037	−2991	20
0.09	0.962815	−4008	26	0.54	0.806046	−2971	21
0.10	0.958807	−3982	+25	0.55	0.803075	−2950	+20
0.11	0.954825	−3957	24	0.56	0.800125	−2930	20
0.12	0.950868	−3933	23	0.57	0.797195	−2910	20
0.13	0.946935	−3910	26	0.58	0.794285	−2890	20
0.14	0.943025	−3384	27	0.59	0.791395	−2870	20
0.15	0.939141	−3857	+23	0.60	0.788525	−2850	+19
0.16	0.935284	−3834	23	0.61	0.785675	−2831	19
0.17	0.931450	−3811	26	0.62	0.782844	−2812	20
0.18	0.927639	−3783	24	0.63	0.780032	−2792	19
0.19	0.923854	−5761	24	0.64	0.777240	−2773	19
0.20	0.920093	−3737	+24	0.65	0.774467	−2754	+19
0.21	0.916356	−3713	24	0.66	0.771713	−2735	19
0.22	0.912643	−3689	25	0.67	0.768978	−2716	19
0.23	0.908954	−3664	23	0.68	0.766262	−2697	18
0.24	0.905290	−3641	25	0.69	0.763565	−2679	18
0.25	0.901649	−3616	+23	0.70	0.760886	−2661	+18
0.26	0.898033	−3593	24	0.71	0.758225	−2643	19
0.27	0.894440	−3569	23	0.72	0.755582	−2624	17
0.28	0.890871	−3546	24	0.73	0.752958	−2607	18
0.29	0.887325	−3522	24	0.74	0.750351	−2589	18
0.30	0.883803	−3498	+22	0.75	0.747762	−2571	+17
0.31	0.880305	−3476	24	0.76	0.745191	−2554	17
0.32	0.876829	−3452	23	0.77	0.742637	−2537	18
0.33	0.873377	−3429	23	0.78	0.740100	−2519	17
0.34	0.869948	−3406	22	0.79	0.737581	−2502	16
0.35	0.866542	−3384	+24	0.80	0.735079	−2486	+19
0.36	0.863158	−3360	22	0.81	0.732593	−2467	16
0.37	0.859799	−3338	23	0.82	0.730126	−2451	16
0.38	0.856461	−3315	22	0.83	0.727675	−2435	16
0.39	0.853146	−3293	23	0.84	0.725240	−2419	17
0.40	0.849853	−3270	+22	0.85	0.722821	−2402	+16
0.41	0.846583	−3248	23	0.86	0.720419	−2386	16
0.42	0.843335	−3225	21	0.87	0.718033	−2370	15
0.43	0.840110	−3204	21	0.88	0.715663	−2355	16
0.44	0.836906	−3183	23	0.89	0.713308	−2339	17

TABLE V (*Continued*)

$\frac{\text{Diameter}}{\text{Length}}$.	K.	Δ_1.	Δ_2.	$\frac{\text{Diameter}}{\text{Length}}$.	K.	Δ_1.	Δ_2.	Δ_3.
0.90	0.710969	− 2322	+ 14	2.50	0.471865	− 9292	+ 405	
0.91	0.708647	− 2308	16	2.60	0.462573	− 8887	378	
0.92	0.706339	− 2292	15	2.70	0.453686	− 8509	355	
0.93	0.704047	− 2277	16	2.80	0.445177	− 8154	330	
0.94	0.701770	− 2261	14	2.90	0.437023	− 7824	312	
0.95	0.699509	− 2247	+ 15	3.00	0.429199	− 7512	+ 293	
0.96	0.697262	− 2232	15	3.10	0.421687	− 7219	275	
0.97	0.695030	− 2217	15	3.20	0.414468	− 6944	260	
0.98	0.692813	− 2202	14	3.30	0.407524	− 6684	245	
0.99	0.690611	− 2188	14	3.40	0.400840	− 6439	230	
1.00	0.688423	−10726	+344	3.50	0.394401	− 6209	+ 220	
1.05	0.677697	−10382	330	3.60	0.388192	− 5989	207	
1.10	0.667315	−10052	316	3.70	0.382203	− 5782	195	
1.15	0.657263	− 9736	303	3.80	0.376421	− 5587	186	
1.20	0.647527	− 9433	290	3.90	0.370834	− 5401	174	
1.25	0.638094	− 9143	+278	4.00	0.365433	− 5227	+ 168	
1.30	0.628951	− 8865	266	4.10	0.360206	− 5059	161	
1.35	0.620086	− 8599	255	4.20	0.355147	− 4898	152	
1.40	0.611487	− 8343	244	4.30	0.350249	− 4746	141	
1.45	0.603144	− 8099	236	4.40	0.345503	− 4605	138	
1.50	0.595045	− 7863	+224	4.50	0.340898	− 4467	+ 134	
1.55	0.587182	− 7639	215	4.60	0.336431	− 4333	125	
1.60	0.579543	− 7424	208	4.70	0.332098	− 4208	118	
1.65	0.572119	− 7216	198	4.80	0.327890	− 4090	115	
1.70	0.564903	− 7018	190	4.90	0.323800	− 3975	102	
1.75	0.557885	− 6828	+184	5.00	0.319825	−18321	+2227	−397
1.80	0.551057	− 6644	176	5.50	0.301504	−16094	1830	−306
1.85	0.544413	− 6468	170	6.00	0.285410	−14264	1524	−241
1.90	0.537945	− 6298	161	6.50	0.271146	−12740	1283	−193
1.95	0.531647	− 6137	154	7.00	0.258406	−11457	1090	−153
2.00	0.525510	−11809	+580	7.50	0.246949	−10367	+ 937	−127
2.10	0.513701	−11229	539	8.00	0.236582	− 9430	810	−104
2.20	0.502472	−10690	499	8.50	0.227152	− 8620	706	− 86
2.30	0.491782	−10191	465	9.00	0.218532	− 7914	620	
2.40	0.481591	− 9726	434	9.50	0.210618	− 7294		
				10.00	0.203324			

TABLE VI

Values of Correction Term A, Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil

(For use in Formula 15.)

$\frac{d}{D}$.	A.	Δ_1.	$\frac{d}{D}$.	A.	Δ_1.
1.00	0.5568	100	0.70	0.2001	144
0.99	0.5468	101	0.69	0.1857	146
0.98	0.5367	103	0.68	0.1711	148
0.97	0.5264	104	0.67	0.1563	150
0.96	0.5160	105	0.66	0.1413	152
0.95	0.5055	106	0.65	0.1261	155
0.94	0.4949	107	0.64	0.1106	157
0.93	0.4842	108	0.63	0.0949	160
0.92	0.4734	109	0.62	0.0789	163
0.91	0.4625	110	0.61	0.0626	166
0.90	0.4515	112	0.60	0.0460	168
0.89	0.4403	113	0.59	0.0292	171
0.88	0.4290	114	0.58	0.0121	174
0.87	0.4176	116	0.57	−0.0053	177
0.86	0.4060	117	0.56	−0.0230	180
0.85	0.3943	118	0.55	−0.0410	184
0.84	0.3825	120	0.54	−0.0594	187
0.83	0.3705	121	0.53	−0.0781	190
0.82	0.3584	123	0.52	−0.0971	194
0.81	0.3461	124	0.51	−0.1165	198
0.80	0.3337	126	0.50	−0.1363	
0.79	0.3211	127			
0.78	0.3084	129	0.50	−0.1363	1053
0.77	0.2955	131	0.45	−0.2416	1178
0.76	0.2824	133	0.40	−0.3594	1335
0.75	0.2691	134	0.35	−0.4928	1542
0.74	0.2557	136	0.30	−0.6471	1823
0.73	0.2421	138	0.25	−0.8294	2232
0.72	0.2283	140	0.20	−1.0526	2877
0.71	0.2143	142	0.15	−1.3403	4054
0.70	0.2001		0.10	−1.7457	

TABLE VII

VALUES OF THE CORRECTION TERM B, DEPENDING ON THE NUMBER OF TURNS OF WIRE ON THE SINGLE LAYER COIL

(For use in Formula 15.)

Number of turns.	B.	Number of turns.	B.
1	0.0000	50	0.3186
2	0.1137	60	0.3216
3	0.1663	70	0.3239
4	0.1973	80	0.3257
5	0.2180	90	0.3270
6	0.2329	100	0.3280
7	0.2443	125	0.3298
8	0.2532	150	0.3311
9	0.2604	175	0.3321
10	0.2664	200	0.3328
15	0.2857	300	0.3343
20	0.2974	400	0.3351
25	0.3042	500	0.3356
30	0.3083	600	0.3359
35	0.3119	700	0.3361
40	0.3148	800	0.3363
45	0.3169	900	0.3364
50	0.3186	1000	0.3365

TABLE VIII

TABLE OF CONSTANTS FOR FORMULA (16)

$\frac{b}{c}$ or $\frac{c}{b}$.	y_1.	y_2.	$\frac{b}{c}$ or $\frac{c}{b}$.	y_1.	y_2.
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	0.54899	0.1269	0.60	0.81823	0.3839
0.10	0.59243	0.1325	0.65	0.82648	0.4274
0.15	0.63102	0.1418	0.70	0.83311	0.4739
0.20	0.66520	0.1548	0.75	0.83831	0.5234
0.25	0.69532	0.1714	0.80	0.84225	0.5760
0.30	0.72172	0.1916	0.85	0.84509	0.6317
0.35	0.74469	0.2152	0.90	0.84697	0.6902
0.40	0.76454	0.2423	0.95	0.84801	0.7518
0.45	0.78154	0.2728	1.00	0.84834	0.8162
0.50	0.79600	0.3066			

TABLE IX

SHAPE-FACTORS, F' AND F'', FOR CERTAIN COIL PROPORTIONS

(For use in Formula 19.)

a.	*b.*	*c.*	*R.*	*F'.*		*F''.*		*F' F''.*	
19	200	2	20	1.008		1.001		1.009	
15	200	10	20		1.015		1.001		1.016
19	100	2	20	1.015		1.003		1.018	
15	100	10	20		1.028		1.002		1.030
19	50	2	20	1.029		1.006		1.035	
15	50	10	20		1.051		1.004		1.055
19	20	2	20	1.064		1.013		1.078	
15	20	10	20		1.097		1.008		1.106
19	12	2	20	1.095		1.019		1.116	
15	12	10	20		1.129		1.011		1.141
19	8	2	20	1.125		1.026		1.154	
15	8	10	20		1.154		1.013		1.169
19	5	2	20	1.163		1.035		1.204	
15	5	10	20		1.182		1.015		1.200
19	4	2	20	1.182		1.040		1.229	
15	4	10	20		1.190		1.016		1.209
19	3	2	20	1.205		1.045		1.259	
15	3	10	20		1.203		1.017		1.223
19	2	2	20	1.235		1.054		1.301	
15	2	10	20		1.216		1.017		1.237
19	1	2	20	1.277		1.065		1.360	
15	1	10	20		1.232		1.018		1.254
19	0.5	2	20	1.302		1.073		1.397	
15	0.5	10	20		1.241		1.018		1.263
19	0.2	2	20	1.320		1.079		1.424	
15	0.2	10	20		1.246		1.019		1.270
19	0.1	2	20	1.326		1.081		1.433	
15	0.1	10	20		1.249		1.019		1.273

SQUARE WINDING-SECTION COILS

17.5	5	5	20	1.172	1.023	1.199
19	2	2	20	1.235	1.054	1.301
19.5	1	1	20	1.292	1.097	1.418
19.75	0.5	0.5	20	1.342	1.163	1.561
19.9	0.2	0.2	20	1.387	1.29	1.79
19.95	0.1	0.1	20	1.407	1.42	2.00

TABLE X

Inductance Per Mile of Complete Metallic Circuit, Formula 35

Size, B. & S.	Diameter in inches.	Distance in inches.	Inductance in mh.	Size, B. & S.	Diameter in inches.	Distance in inches.	Inductance in mh.
0000	0.46	12	2.707	4	0.204	12	3.231
		18	2.969			18	3.492
		24	3.154			24	3.678
		48	3.600			48	4.124
000	0.41	12	2.782	5	0.182	12	3.305
		18	3.043			18	3.566
		24	3.228			24	3.751
		48	3.675			48	4.198
00	0.365	12	2.857	6	0.162	12	3.380
		18	3.118			18	3.641
		24	3.303			24	3.826
		48	3.750			48	4.273
0	0.325	12	2.930	7	0.144	12	3.456
		18	3.193			18	3.717
		24	3.378			24	4.349
		48	3.824			48	4.349
1	0.289	12	3.007	8	0.128	12	3.531
		18	3.268			18	3.792
		24	3.453			24	3.978
		48	3.900			48	4.424
2	0.258	12	3.080	9	0.114	12	3.606
		18	3.341			18	3.867
		24	3.526			24	4.052
		48	3.973			48	4.500
3	0.229	12	3.157	10	0.102	12	3.678
		18	3.418			18	3.940
		24	3.603			24	4.124
		48	4.050			48	4.571

CHAPTER III

CAPACITY

THE capacity of a conductor is defined as the ratio of its charge to its potential, when all other conductors are at zero potential.

$$C = \frac{Q}{E} \tag{1}$$

or, according to Maxwell, the capacity of a conductor is its charge when its own potential is unity and that of all the other conductors is zero.

The capacity between two conductors is the ratio of the charge on either conductor to the difference of potential between the conductors, assuming also that all other conductors are at zero potential, otherwise the capacity is considerably influenced to an extent depending on the proximity of the other conductors.

In general the capacity of a conductor depends on the geometrical configuration of its own surface, its distance from other conductors and the material medium between them.

The problem of determining the capacity of a conductor, or a system of conductors, offers considerable mathematical difficulties, and there are not therefore as many formulæ available for the calculation of capacities as in the case of inductances. For linear conductors, however, which are the most important in practice, a sufficient number of formulæ have been derived to enable us to compute the capacity of any arrangement of circuits. Owing to the extreme importance of the electrical constants of linear conductors in problems of transmission of electrical energy, we shall give here an extensive list of such formulæ.

The unit of capacity is the capacity of a condenser in which unit charge produces a unit difference of potential between its conductors, which may be either in electrostatic or electromagnetic units. The ratio of the electrostatic to the electromagnetic unit of capacity is equal to V^2 where $V = 3 \times 10^{10}$ cm. per sec. (velocity of light);

hence

$$C_m = \frac{1}{V^2} C_e. \tag{2}$$

The practical unit of capacity is the farad. It is the capacity of a condenser which has a potential difference of one volt when charged with one coulomb.

$$1 \text{ farad} = 10^{-9} \text{ C.G.S. units.}$$
$$1 \text{ microfarad} = 10^{-15} \text{ C.G.S. units.}$$

In practice the microfarad is generally used.

If C is the capacity in microfarads and C_e the capacity in electrostatic units, we have

$$C = \frac{10^{15}}{V^2} C_e = \frac{1}{9 \times 10^5} C_e.$$

Parallel or Series Arrangement. — In electrical circuits we frequently use a number of condensers which may be either connected in parallel or in series. For the parallel arrangement the total capacity is the sum of the capacities of the several condensers, that is,

$$C = C_1 + C_2 + C_3 + \cdots + C_n. \tag{3}$$

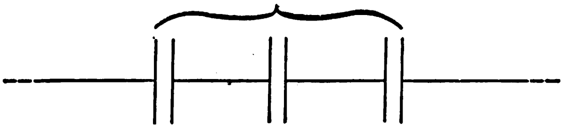

FIG. 25.

In the series arrangement the resultant capacity is given by the following expression:

$$\left.\begin{aligned} \frac{C}{1} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}, \\ \text{or} \quad C &= \frac{C_1C_2C_3 \ldots C_n}{C_2C_3C_4 \ldots C_n + C_1C_3C_4 \ldots C_n + \ldots C_1C_2 \ldots C_{n-1}}. \end{aligned}\right\} \quad (4)$$

The capacity of three condensers in series for instance is

$$C = \frac{C_1C_2C_3}{C_2C_3 + C_1C_3 + C_1C_2}. \tag{5}$$

The total capacity of several condensers connected in series is always less than the capacity of the smallest condenser in the series.

Example. — Three condensers of capacities C_1, C_2, C_3 connected in series.

Let $C_1 = 0.01$ mf., $C_2 = 0.05$ mf., $C_3 = 0.1$ mf.

$$C = \frac{0.01 \times 0.05 \times 0.1}{0.005 + 0.001 + 0.0005} = \frac{0.00005}{0.0065} = 0.0077 \text{ mf.},$$

which is less than the value of C_1, that of the smallest condenser.

Capacity of Parallel Plates. — If the area of each plate is denoted by A sq. cm. and the distance between them by h cm., the capacity of the condenser is

$$\left.\begin{aligned} C &= \frac{A}{4\pi h} \\ &= \frac{A}{4\pi h} \, \frac{1}{9 \times 10^5} \text{ mf.} \end{aligned}\right\} \qquad (6)$$

For circular plates of radius r, we have

$$C = \frac{r^2}{4h} \, \frac{1}{9 \times 10^5} \text{ mf.} \qquad (7)$$

If the space between the plates is filled with a uniform dielectric of specific inductive capacity K, then

$$C = \frac{AK}{4\pi h} \, \frac{1}{9 \times 10^5} \text{ mf.} \qquad (8)$$

Formulæ (6) to (8) are only approximate; they were derived on the assumption that the distribution of the electric force between the plates is uniform throughout the entire area. This, however, is not the case; the lines of force curve around the edges, and the correction due to this effect cannot be neglected in accurate calculations. This correction factor has been determined only for circular plates, and the corrected formula for this case is as follows:*

$$C = \frac{r^2}{4h} \left\{ 1 + \frac{h}{\pi r} \left[\log_e \frac{16\pi(h+t)}{h^2} + \frac{t}{h} \log_e \frac{h+t}{t} + 1 \right] \right\} \frac{1}{9 \times 10^5} \text{ mf.,} \qquad (9)$$

where t is thickness of the plates and h is the distance between them.

For very thin plates

$$C = \frac{r^2}{4h} \left\{ 1 + \frac{h}{\pi r} \left[\log_e \frac{16\pi}{h} + 1 \right] \right\} \frac{1}{9 \times 10^5} \text{ mf.} \qquad (10)$$

---

* G. Kirchoff, Gesammelte Abhandlungen, p. 112, "Zur Theorie des Condensators." See also J. A. Fleming, "The Principles of Electric Wave Telegraphy," p. 176.

Example. — Capacity of two-plate circular condenser of the following dimensions.

$r = 10$ cm., $h = 0.2$ cm., $t = 0.05$ cm. air dielectric, $K = 1$.

By formula (7),

$$C = 13.9 \times 10^{-5} \text{ mf.}$$

By formula (9),

$$C = 14.51 \times 10^{-5} \text{ mf.}$$

By formula (10),

$$C = 14.44 \times 10^{-5} \text{ mf.}$$

It is seen that in this particular case the capacity as computed by formula (7) is in error by about 4.5 per cent. The smaller the area of the plates and the greater the distance between them, the larger is the correction term.

For a multiple-plate condenser consisting of n plates, the capacity is $(n - 1)$ times the expression for capacity given in formulæ (6) to (8).

When the dielectric of a plate condenser consists of several layers of different specific inductive capacities, we have, on neglecting the correction factor,

$$C = \frac{A}{4\pi\left\{\frac{h_1}{K_1} + \frac{h_2}{K_2} + \cdots\right\}} \frac{1}{9 \times 10} \text{ mf.,} \tag{11}$$

where h_i and K_i denote the thickness and specific inductive capacity of layer i, and n is the number of layers, or

$$C = \frac{A}{4\pi \sum_{i=1}^{i=n} \frac{h_i}{K_i}} \frac{1}{9 \times 10^5} \text{ mf.} \tag{12}$$

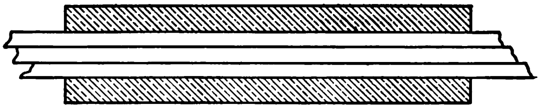

FIG. 26.

Example. — Same constants as in the preceding example.

Two-plate circular condenser,

$r = 10$ cm., $A = \pi \times 100$,
$h_1 = h_2 = h_3 = h_4 = 0.05$ cm.,
h (distance between plates) $= 0.2$ cm.,
$K_1 = 1.5$, $K_2 = 2$, $K_3 = 2.5$, $K_4 = 3$.

$$C = \frac{\pi \times 100}{4\pi\left\{\frac{0.05}{1.5}+\frac{0.05}{2}+\frac{0.05}{2.5}+\frac{0.5}{3}\right\}}\frac{1}{9\times 10^5} = 29 \times 10^{-6}\,\text{mf.}$$

Capacity of Concentric Spheres.—

$$C = \frac{r_1 r_2}{r_1 - r_2}\frac{1}{9 \times 10^5}\,\text{mf.} \tag{13}$$

r_2 is the inner radius of outside sphere and r_1 is the radius of inner sphere. This formula is exact, no correction term is required.

When r_2 is infinitely large

$$C = \frac{r_1}{9 \times 10^5}\,\text{mf.}, \tag{14}$$

which is the capacity of an isolated sphere in infinite space.

From formula (14) it is obvious that a sphere will have to be of radius 9 × 10^5 cm., or 9 km., to have a capacity of one microfarad.

When the centers of the spheres are separated by a small distance d, the capacity is given by the following expression:

$$C = \frac{r_1 r_2}{r_1 - r_2}\left\{1 - \frac{r_1 r_2 d^2}{(r_1 - r_2)(r_1^3 - r_2^3)}\right\}. \tag{15}$$

If the dielectric between the spheres is made up of several spherical layers of different specific inductive capacities, the expression for the capacity is as follows:

$$C = \frac{1}{\sum_{i=1}^{i=n}\frac{1}{K_i}\left(\frac{1}{r_{i-1}} - \frac{1}{r_i}\right)}; \tag{16}$$

r_i is the radius of the sphere which separates the layer i from the layer $i + 1$.

Comparing formulæ (13) and (15) it is seen that displacing the centers of the spheres reduces the capacity.

Example.—

$r_1 = 30$ cm., $r_2 = 28$ cm.

By formula (13),

$$C = \frac{30 \times 28}{30 - 28}\frac{1}{9 \times 10^5} = 420\frac{1}{9 \times 10^5}\,\text{mf.}$$

If the centers of the spheres are separated by 0.5 cm., that is, in formula (16) $d = 0.5$, we get

$$C = \frac{30 \times 28}{30 - 28} \left\{ 1 - \frac{210}{10,096} \right\}$$

$$= 420 \{1 - 0.021\} \frac{1}{9 \times 10^5} = 411 \frac{1}{9 \times 10^5} \text{ mf.}$$

Displacing the centers of the spheres by 0.5 cm. reduces the capacity by two per cent approximately.

Capacity of Disk insulated in Free Space. —

$$C = \frac{d}{\pi}, \tag{17}$$

d = diameter of disk in centimeters.

Capacity Coefficients of Two Spheres.* — If we apply a difference of potential to two spherical conductors which are separated from each other, the relation between the charges and potentials are given by the following equation:

$$q_1 = C_1V_1 + C_{12}V_2 \quad \text{and} \quad q_2 = C_2V_2 + C_{12}V_1, \tag{18}$$

where C_1, C_2 and C_{12} are the capacity coefficients of the spheres.

Maxwell† derived general formulæ for the capacity coefficients C_1, C_2 and C_{12} covering the cases of equal and unequal radii of the two spheres with any distance separation between them. The general formulæ, however, as given by Maxwell are somewhat complicated, so we shall give here only the simplified formulæ obtained by Russell, which are applicable only for spheres of equal radii.

Let the radius of each sphere be a and d the distance between their centers; then,

$$\left. \begin{aligned} C_1 &= \lambda \sum_{s=0}^{s=\infty} \frac{1}{\sinh (2s + 1) \alpha} \\ -C_{12} &= \lambda \sum_{s=0}^{s=\infty} \frac{1}{\sinh 2s\alpha}, \end{aligned} \right| \tag{19}$$

where

$$\sinh \alpha = \frac{\lambda}{a} \quad \text{and} \quad 4\lambda^2 = d^2 - 4a^2. \tag{20}$$

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* Alexander Russell, *Journal of the Institution of Electrical Engineers*, Feb., 1912.

† Maxwell, *Electricity and Magnetism*, Vol. 1, p. 270, § 173.

Denoting $\frac{d}{a}$ by y, formulæ (19) may be put in the following form:

$$\left.\begin{aligned} \frac{C_1}{a} &= 1 + \frac{1}{y^2} + \frac{2}{y^4} + \frac{5}{y^6} + \frac{15}{y^8} + \frac{49}{y^{10}} + \frac{166}{y^{12}} + \cdots \\ &= 1 + \frac{1}{y^2} + \frac{2y^2+1}{(y^3+y)^2 - (2y^2+1)^2} - \frac{1}{y^8}, \\ \text{and} \qquad -\frac{C_{12}}{a} &= \frac{1}{y} + \frac{y^3+y}{(y^3+y)^2 - (2y^2+1)^2}; \end{aligned}\right\} \qquad (21)$$

also

$$\left.\begin{aligned} -\frac{\partial C_1}{\partial d} &= A = \frac{2}{y^3 - 4y} - \frac{2}{y(y^3 - 2y)^2}, \\ \frac{\partial C_{12}}{\partial d} &= B = \frac{1}{y^2 - 4} - \frac{y^2+1}{y^6} - \frac{1}{y^8 - 4y^6}. \end{aligned}\right\} \qquad (22)$$

C_1 is the capacity of the sphere when all neighboring conductors are at zero potential. When the charges on the two spheres are $+q$ and $-q$ respectively, and V is the difference of potential between them, then $\frac{q}{V}$ is the capacity C between the spheres and this is the capacity generally considered in engineering problems.

In this case, if V and $-V$ are the potentials of the two spheres,

$$\frac{q}{2V} = C = \frac{C_1 - C_{12}}{2}. \qquad (23)$$

When both spheres and all neighboring conductors are at the same potential V, then

$$\left.\begin{aligned} q &= C_1 V + C_{12} V. \\ \frac{q}{V} &= C_1 + C_{12} = C'. \end{aligned}\right\} \qquad (24)$$

When y is not less than 3 the values of C and C' are given by the following formulæ:

$$\left.\begin{aligned} \frac{2C}{a} &= \frac{y(y-1)^2}{y(y-1)^2 - 1} + \frac{y+1}{y^2} - \frac{1}{y^8}, \\ \frac{C'}{a} &= \frac{y(y+1)^2}{y(y+1)^2 + 1} - \frac{y-1}{y^2} - \frac{1}{y^8}. \end{aligned}\right\} \qquad (25)$$

For values of y greater than 8 we have the following formulæ:

$$\left.\begin{aligned}\frac{2C}{a} &= 1+\frac{1}{y}+\frac{1}{y^2}+\frac{1}{y^3}+\frac{2}{y^4},\\ \frac{C'}{a} &= 1-\frac{1}{y}+\frac{1}{y^2}-\frac{1}{y^3}+\frac{2}{y^4}.\end{aligned}\right\}\qquad(26)$$

In Table XI, p. 122, are given the calculated values of $\frac{C}{a}$ and $\frac{C'}{a}$ for values of $\frac{d}{a}$ from 2 to 1000.

The Laws of Attraction and Repulsion Between the Spheres When the Potentials are Given. — If we denote by W the electrostatic energy of the system, we have

$$W = \tfrac{1}{2}C_1(V_1^2+V_2^2)+C_{12}V_1V_2 \qquad (27)$$

and the force acting between the spheres is

$$F = \frac{\partial W}{\partial d} = -\frac{1}{2}A\,(V_1^2+V_2^2)+BV_1V_2. \qquad (28)$$

A and B are given by equation (22).

In practice three cases may arise:

(1) When the potentials are equal and opposite.
(2) When they are equal.
(3) When one of the spheres is at zero potential.

In the first case the force F is attractive and is given by

$$F=(A+B)\,V^2=\left(\frac{1}{y^2-2y}-\frac{1}{y^4}-\frac{(1+y)^2}{y^8}-4\frac{2y^2+y+6}{y^{11}}\right)V^2. \qquad (29)$$

In the second case the force F' is repulsive.

$$F'=(B-A)V^2=\left(\frac{1}{y^2+2y}-\frac{1}{y^4}-\frac{(1-y)^2}{y^8}+4\frac{2y^2-y+6}{y^{11}}\right)V^2. \qquad (30)$$

In the third case the force F'' is attractive.

$$F''=\frac{1}{2}AV^2=\left(\frac{1}{y^3-4y}-\frac{1}{y\,(y^3-2y)^2}\right)V^2. \qquad (31)$$

When y is less than 2.001 we have the following very approximate formulæ:

$$\left.\begin{aligned} F &= \frac{aV^2}{2(d-2a)}, \\ F' &= 0.07386\,V^2. \end{aligned}\right\} \tag{32}$$

For values of $\frac{d}{a}$ greater than 10, the following formulæ give a four-figure accuracy.

$$\left.\begin{aligned} F &= \frac{a^2}{d^2}\frac{1}{1-\frac{2a}{d}}V^2, \\ F' &= \frac{a^2}{d^2}\frac{1}{1+\frac{2a}{d}}V^2. \end{aligned}\right\} \tag{33}$$

Capacity of Linear Conductors. — For the transmission of electrical energy, linear conductors are employed in the form of overhead wires, concentric mains or cables. In some of these cases the derivation of capacity formulæ offers considerable analytical difficulties. A sufficient number of formulæ have however been derived to cover nearly all cases which are apt to arise in practice.

Overhead Wires.* — We shall first consider the case of overhead lines where the ground acts as the return conductor, which is generally the case for telegraph lines.

If we have a single wire of radius a suspended at height h above ground its capacity is

$$C = \frac{l}{2\log_e\frac{2h}{a}}\frac{1}{9\times 10^5}\text{ mf.} \tag{34}$$

l is the length of the line in centimeters; h and a can be expressed in any units so long as they are both expressed in the same units.

We may write formula (34) in the following form:

$$C = \frac{0.0894}{\log_e\frac{2h}{a}}\text{ mf. per mile.} \tag{35}$$

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* See O. Heaviside, *Collected Papers*, Vol. I, pp. 42–46.

F. F. Fowle, "Calculation of Capacity Coefficients of Wires," *Electrical World*, Vol. 58, Aug., 1911.

Formulæ (34) and (35) give only an approximation, the degree of approximation being better the greater the ratio $\frac{h}{a}$. For large values of $\frac{h}{a}$ formula (34) gives fairly accurate results. Prof. Kennelly * has obtained an exact formula for this case which is as follows:

$$C = \frac{l}{2 \cosh^{-1} \frac{h}{a}} \frac{1}{9 \times 10^5} \text{mf.}, \tag{36}$$

or we may put it,

$$C = \frac{0.0894}{\cosh^{-1} \frac{h}{a}} \text{ mf. per mile.} \tag{37}$$

Example. —

$$\frac{h}{a} = 5.$$

By (35),

$$C = \frac{0.0894}{\log_e 10} = 0.0384 \text{ mf. per mile.}$$

By (37),

$$C = \frac{0.0894}{2.2924} = 0.0390 \text{ mf. per mile.}$$

The difference in results by the two formulæ is about 1.5 per cent. For greater values of $\frac{h}{a}$ the results by the two formulæ will be more nearly equal.

Formula (35) was derived on the assumption that the conductor is suspended in free space and that there are no other conductors near by to influence the value of its capacity. This is a condition which is never realized in practice. Generally there are a number of wires suspended on the same pole and the capacity of each conductor is affected by the presence of the other conductors. In the following we shall give some formulæ for the case of two or more grounded wires suspended on the same pole.

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* A. E. Kennelly, *Proc. Am. Phil. Soc.*, Vol. 48, pp. 142–165, 1909.

Two Grounded Wires. — Two wires 1 and 2 are suspended at heights h_1 and h_2 above ground and at distance d_{12} from each other (see Fig. 27).

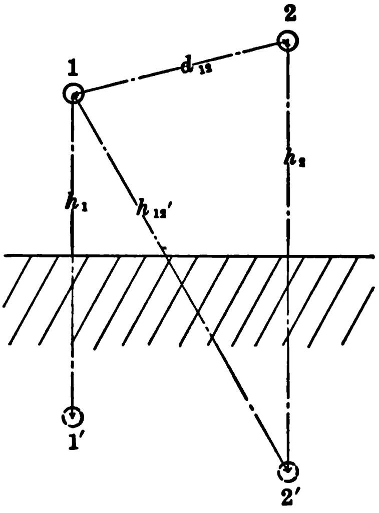

Fig. 27.

The capacities of wires 1 and 2 and the mutual capacity between them are

$$\left.\begin{aligned} C_1 &= \frac{0.0894 \log_e \dfrac{2\,h_2}{a_2}}{D} \text{ mf. per mile,} \\ C_2 &= \frac{0.0894 \log_e \dfrac{2\,h_1}{a_1}}{D} \text{ mf. per mile,} \\ -\,C_{12} &= \frac{0.0894 \log_e \dfrac{h_{12}}{d_{12}}}{D} \text{ mf. per mile,} \end{aligned}\right\} \qquad (38)$$

where
$$D = \left(\log_e \frac{2\,h_1}{a_1}\right)\left(\log_e \frac{2\,h_2}{a_2}\right) - \left(\log_e \frac{h_{12}}{d_{12}}\right)^2.$$

a_1 and a_2 are the radii of wires 1 and 2 respectively.

When the wires are of the same radii, $a_1 = a_2 = a$, and at the same heights above ground, $h_1 = h_2 = h$, formulæ (38) reduce to

$$\left.\begin{aligned} C_1 = C_2 &= \frac{0.0894 \log_e \dfrac{2h}{a}}{\left(\log_e \dfrac{2h}{a}\right)^2 - \left(\log_e \dfrac{\sqrt{d^2+4h^2}}{d}\right)^2} \text{ mf. per mile,} \\ -C_{12} &= \frac{0.0894 \log_e \dfrac{\sqrt{d^2+4h^2}}{d}}{\left(\log_e \dfrac{2h}{a}\right)^2 - \left(\log_e \dfrac{\sqrt{d^2+4h^2}}{d}\right)^2} \text{ mf. per mile.} \end{aligned}\right\} \quad (39)$$

Example. — Single wire, No. 8 B. & S., suspended 25 feet above ground.

$$a = 0.0642 \text{ in.}, \quad \log_e \frac{2h}{a} = \log_e \frac{50 \times 12}{0.0642} = 9.142.$$

$$C = \frac{0.0894}{9.142} = 0.00978 \text{ mf. per mile.}$$

Example. — Two grounded wires, No. 8 B. & S., equal heights above ground 25 feet and distance apart 1 foot.

As in the preceding example we have

$$\log_e \frac{2h}{a} = 9.142, \qquad \log_e \frac{\sqrt{d^2+4h^2}}{d} = 3.912.$$

$$C_1 = C_2 = \frac{0.0894 \times 9.142}{(9.142)^2 - (3.912)^2} = 0.0120 \text{ mf. per mile,}$$

$$-C_{12} = \frac{0.0894 \times 3.912}{(9.142)^2 - (3.912)^2} = 0.0052 \text{ mf. per mile.}$$

An additional wire on the same pole increases the individual capacity of each wire. In the case considered in above examples, the increase in capacity is about 23 per cent.

As we increase the number of wires on the same pole the individual capacities of the several wires are continually increased. We shall give here formulæ for the capacities of three and four wires. Though the formulæ are not very simple, and are unwieldy for numerical computation, yet the numerical examples worked out here will give some idea of the magnitude of the increase in capacities as the number of wires is being increased.

Three Grounded Wires. — The wires are at equal heights h above ground, the same distances d apart and of equal radii a. Wires 1 and 3 are symmetrically situated with respect to wire

2, hence their capacities are equal and for the same reason the mutual capacities C_{12} and C_{23} are equal.

$$\left.\begin{aligned}
C_1 = C_3 &= \frac{0.0894\left\{\left(\log_e \frac{2h}{a}\right)^2 - \left(\log_e \frac{\sqrt{d^2+4h^2}}{d}\right)^2\right\}}{D} \text{ mf. per mile,}\\
C_2 &= \frac{0.0894\left\{\left(\log_e \frac{2h}{a}\right)^2 - \left(\log_e \frac{\sqrt{d^2+h^2}}{d}\right)^2\right\}}{D} \text{ mf. per mile,}\\
C_{12} = C_{23} &= \\
&\frac{0.0894\left\{\left(\log_e \frac{\sqrt{d^2+4h^2}}{d}\right)\left(\log_e \frac{\sqrt{d^2+h^2}}{d} - \log_e \frac{2h}{a}\right)\right\}}{D} \text{ mf. per mile,}\\
C_{13} &= \\
&\frac{0.0894\left\{\left(\log_e \frac{\sqrt{d^2+4h^2}}{d}\right)^2 - \log_e \frac{2h}{a} \times \log_e \frac{\sqrt{d^2+4h^2}}{d}\right\}}{D} \text{ mf. per mile,}
\end{aligned}\right\} (40)$$

where

$$D = \left(\log_e \frac{2h}{a}\right)^3 + 2\log_e \frac{\sqrt{d^2+h^2}}{d}\left(\log_e \frac{\sqrt{d^2+4h^2}}{d}\right)^2$$
$$- 2\log \frac{2h}{a}\left(\log_e \frac{\sqrt{d^2+4h^2}}{d}\right)^2 - \log_e \frac{2h}{a}\left(\log_e \frac{\sqrt{d^2+h^2}}{d}\right)^2.$$

Example. — Use same constants as in preceding examples.
$a = 0.0642$ in., $h = 25$ feet, $d = 1$ ft.

$$\log_e \frac{2h}{a} = 9.142, \log_e \frac{\sqrt{d^2+4h^2}}{d} = 3.912, \quad \log_e \frac{\sqrt{d^2+h^2}}{d} = 3.219.$$

$$D = 764.5 + 98.54 - 279.84 - 94.80 = 488.4.$$

$$C_1 = C_3 = \frac{0.0894\{(9.142)^2 - (3.912)^2\}}{488.4} = 0.0125 \text{ mf. per mile.}$$

$$C_2 = \frac{0.0894\{(9.142)^2 - (3.22)^2\}}{488.4} = 0.0134 \text{ mf. per mile.}$$

$$C_{12} = C_{23} = \frac{0.0894\{3.912 \times 5.92\}}{488.4} = -\ 0.0043 \text{ mf. per mile.}$$

$$C_{13} = \frac{0.0894\{(3.912)^2 - 9.142 \times 3.912\}}{488.4} = -\ 0.0038 \text{ mf. per mile.}$$

Four Grounded Wires. — Suppose we have four grounded wires placed at the corner of a rectangle as shown in Fig. 28, and we shall assume that the wires are of equal radii r.

Let

$$\log_e \frac{h_{11}}{r} = a, \qquad \log_e \frac{h_{12}}{d_{12}} = b,$$
$$\log_e \frac{h_{13}}{d_{13}} = c, \qquad \log_e \frac{h_{14}}{d_{14}} = d,$$
$$\log_e \frac{h_{33}}{r} = e, \qquad \log_e \frac{h_{34}}{d_{12}} = f.$$

The significance of the subscripts to h and d are obvious from the figure.

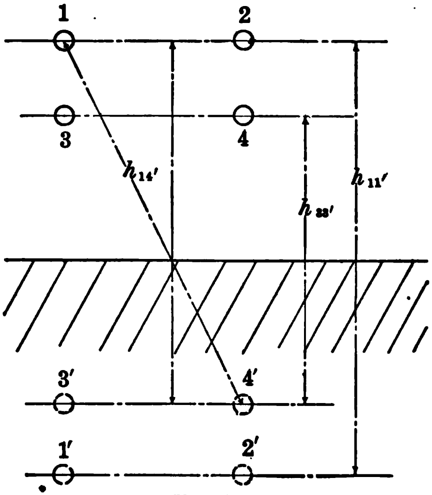

FIG. 28.

The capacities of the separate wires and the mutual capacities are as follows:

$$\left.\begin{aligned}
C_1 = C_2 &= \{a(e^2 - f^2) + d(cf - de) - c(df - ce)\} \div D,\\
C_3 = C_4 &= \{e(a^2 - b^2) + d(bc - ad) + c(bd - ac)\} \div D,\\
-C_{12} &= \{d(df - ce) + c(cf - de) + b(e^2 - f^2)\} \div D,\\
-C_{24} = -C_{13} &= \{c(c^2 - d^2) + b(de - cf) + a(df - ce)\} \div D,\\
-C_{23} = -C_{14} &= \{b(df - ce) + a(de - cf) + d(c^2 - d^2)\} \div D,\\
-C_{34} &= \{d(bd - ac) + c(bc - ad) + f(a^2 - b^2)\} \div D,\\
D = 2\{(a^2 - b^2)(e^2 - f^2) &+ (c^2 - d^2)^2 + 2(ac - bd)(df - ec)\\
&+ 2(ad - bc)(cf - de)\}.
\end{aligned}\right\} \quad (41)$$

Example. — The wires are of No. 8 B. & S. $r = 0.0642$ in.

The height of the lower two wires is 25 ft. above ground.

The vertical distance between wires $d_{13} = 1$ ft.

The horizontal distance between wires $d_{12} = 2$ ft.

We have then

$$h_{11} = 52 \text{ ft.}, \quad h_{12} = \sqrt{2708}, \quad h_{13} = 51, \quad h_{14} = \sqrt{2605},$$

$$h_{33} = 50, \quad h_{34} = \sqrt{2504}, \quad d_{12} = 2, \quad d_{13} = 1, \quad d_{14} = \sqrt{5},$$

$$a = 9.182, \quad b = 3.259, \quad c = 3.932, \quad d = 2.628, \quad e = 9.142, \quad f = 3.22.$$

$$D = 2\,[73.65 \times 73.17 + (8.52)^2 - 2 \times 2.75 \times 27.45 - 2 \times 11.33 \times 11.38] = 7388,$$

$$C_1 = C_2 = \frac{671.7 - 29.93 + 107.88}{7388} \times \frac{1.61}{9} = 0.0181 \text{ mf. per mile},$$

$$C_3 = C_4 = \frac{673.16 - 29.80 - 108.1}{7388} \times \frac{1.61}{9} = 0.013 \text{ mf. per mile},$$

$$-C_{12} = \frac{-72.2 - 44.7 + 238.5}{7388} \times \frac{1.61}{9} = 0.0029 \text{ mf. per mile},$$

$$-C_{24} = -C_{13} = \frac{33.5 + 37.1 - 252}{7388} \times \frac{1.61}{9} = 0.0044 \text{ mf. per mile},$$

$$-C_{23} = -C_{14} = \frac{-89.5 + 104.5 + 22.4}{7388} \times \frac{1.61}{9} = 0.0009 \text{ mf. per mile},$$

$$-C_{34} = \frac{-72.3 - 44.5 + 237.1}{7388} \times \frac{1.61}{9} = 0.0029 \text{ mf. per mile}.$$

Capacity of Condenser formed by Two Long Parallel Cylinders.* — The capacity of a looped conductor may be expressed either as the capacity per unit length of each wire or the capacity per unit length of loop, the capacity of the latter being one-half that of the former. In single-phase transmission lines there is not much advantage in favor of one or the other, but in the case of three-phase transmission lines it is of some advantage to express the capacity per unit length of each wire.

* Alex. Russell, "The Theory of Alternating Currents," Vol. I, Chapters IV and V.

"Theorie der Wechselströme," J. L. LaCour und O. S. Bragstad, pp. 588–602.

H. Pender and H. S. Osborne, "The Capacity between Two Equal Parallel Wires," *Electrical World*, Vol. 56, pp. 667–670, Sept., 1910.

A. E. Kennelly, "Graphical Representation of the Electrostatic Capacity between Wires," *Electrical World*, Vol. 56, pp. 1000–1002, Oct., 1910.

In formulæ (42) to (48) we shall employ the following notations:

a = radius of conductor,
d = interaxial distance,
h = height above ground.

When the interaxial distance between wires is large compared with the radius of the wire, we have for the capacity per unit length of conductor

$$C = \frac{1}{2 \log_e \frac{d}{a}} \tag{42}$$

and the capacity per unit length of looped conductor

$$C = \frac{1}{4 \log_e \frac{d}{a}}. \tag{43}$$

Expressing the capacity in mf. per mile, formulæ (42) and (43) assume the forms

$$\left.\begin{aligned} C &= \frac{0.0894}{\log_e \frac{d}{a}}, \\ C &= \frac{0.0447}{\log_e \frac{d}{a}}. \end{aligned}\right\} \tag{44}$$

When the interaxial distance is not very large, we must use the more accurate formula

$$C = \frac{0.0894}{\log_e \left\{ \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \right\}} \text{ mf. per mile of single wire.} \tag{45}$$

In the derivation of formula (45) the influence of the earth, as a near-by conducting plane, was neglected. Its effect is negligible where the wires are at a considerable height above ground, which is usually the case for transmission lines. Some cases, however, may arise where it is desirable to take into account the influence of the earth and in that case, if we have two wires in the same horizontal plane height h above ground, the capacity of each wire is

$$C = \frac{0.0894}{\log_e \left\{ \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \right\} - \log_e \sqrt{1 + \left(\frac{d}{2h}\right)^2}} \tag{46}$$

mf. per mile of single wire.

If the two wires are placed one vertically above the other, the capacity is given by

$$C = \frac{0.0894}{\log_e \left\{ \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \right\} + \log_e \left(1 - \frac{d^2}{(d+2h)^2}\right)} \text{ mf. per mile of single wire.} \quad (47)$$

If d is small compared with h, equation (47) may be written

$$C = \frac{0.0894}{\log_e \left\{ \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \right\} - \frac{2d^2}{(d+2h)^2}} \text{ mf. per mile,} \quad (48)$$

where h, in formulæ (47) and (48), is the height of the lower wire above ground.

The capacity of the conductors is a little less when they are placed one above the other than when they are placed in the same horizontal plane, assuming that the distance between the wires is the same in each case.

Example. —

$$a = 0.1 \text{ in.}, \qquad d = 2 \text{ ft.}, \qquad h = 5 \text{ ft.},$$

$$\log_e \left\{ \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \right\} = \log_e 240 = 5.480,$$

$$\log_e \left(1 + \frac{d^2}{4h^2}\right) = \log_e 1.04 = 0.039, \quad \log_e \left(1 - \frac{d^2}{(d+2h)^2}\right) = -\,0.028.$$

By formula (46), $C = \frac{0.0894}{5.48 - 0.039} = 0.01643$ mf. per mile.

By formula (47), $C = \frac{0.0894}{5.48 - 0.028} = 0.01640$ mf. per mile.

It is seen from the above example that even when the conductors are brought within 5 feet from the earth the capacity is increased only by about 0.7 per cent, so that in all practical cases we can neglect the earth's influence and use the simple formula (44) or (45).

When the radii of the wires are not equal, as has been assumed in the above formulæ, we have

$$C = \frac{0.0894}{\frac{1}{2} \log_e \left\{ \alpha + \sqrt{\alpha^2 - 1} \right\}} \text{ mf. per mile single wire,} \quad (49)$$

where $$\alpha = \frac{d^2 - a_1^2 - a_2^2}{2\,a_1 a_2}.$$

a_1 and a_2 are the radii of the two wires.

The capacities of looped conductors can be also expressed very conveniently in terms of anti-hyperbolic functions, as has been done by Prof. Kennelly.*

When the radii of the wires are equal we have

$$C = \frac{0.0894}{\cosh^{-1}\left(\frac{d}{2\,a}\right)} \text{ mf. per mile of single wire.} \tag{50}$$

This formula is equivalent to (45).

For unequal radii we have

$$C = \frac{0.0894}{\frac{1}{2}\left(\cosh^{-1}\frac{d}{2\,a_1} + \cosh^{-i}\frac{d}{2\,a_2}\right)} \tag{51}$$

mf. per mile of single wire, which is equivalent to (49).

Formulæ (45) and (50) are equivalent to each other since

$$\log_e \left\{\frac{d}{2\,a} + \sqrt{\left(\frac{d}{2\,a}\right)^2 - 1}\right\} = \cosh^{-1}\frac{d}{2\,a}$$

and both formulæ are exact.

In Table XII, p. 122, the values of $\log_e \frac{d}{a}$ and $\cosh^{-1}\frac{d}{2\,a}$ are given for values of $\frac{d}{2\,a}$ from 1.01 to 25.

Formula (44) gives fairly accurate results when $\frac{d}{2\,a}$ is greater than 5, increasing in degree of accuracy as the value of $\frac{d}{2\,a}$ increases. When $\frac{d}{2\,a}$ is small, that is, when the wires are close together, the results by formula (44) are considerably in error.

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* A. E. Kennelly, *Proc. Am. Phil. Socy.*, Vol. 48, pp. 142–165, 1909.

Example. —

For $$\frac{d}{2a} = 10.$$

By formula (44),

$$C = \frac{0.0894}{2.9956} = 0.0298 \text{ mf. per mile.}$$

By formula (45) or (50),

$$C = \frac{0.0894}{2.9932} = 0.0299 \text{ mf. per mile.}$$

For $$\frac{d}{2a} = 1.5.$$

By formula (44),

$$C = \frac{0.0894}{1.0986} = 0.0814 \text{ mf. per mile.}$$

By formula (45) or (50),

$$C = \frac{0.0894}{0.8670} = 0.1031 \text{ mf. per mile.}$$

The error in formula (44) for this case is about 27 per cent.
Example. — For wires of different diameters,

$$a_1 = 0.5 \text{ cm.}, \qquad a_2 = 1 \text{ cm.}, \qquad d = 10 \text{ cm.}$$

By formula (49),

$$\alpha = \frac{100 - 1 - 0.25}{1} = 98.75,$$

$$\log_e \{\alpha + \sqrt{\alpha^2 - 1}\} = 5.275,$$

$$C = \frac{0.0894}{\frac{1}{2}(5.275)} = 0.034 \text{ mf. per mile single wire.}$$

Working out the same example by formula (50) we get

$$\cosh^{-1}\frac{d}{2a_1} = \cosh^{-1}\frac{10}{1.0} = 2.9932,$$

$$\cosh^{-1}\frac{d}{2a_2} = \cosh^{-1}\frac{10}{2} = 2.924,$$

$$C = \frac{0.0894}{\frac{1}{2}(2.9932 + 2.924)} = 0.034 \text{ mf. per mile single wire.}$$

The results by formulæ (49) and (50) agree exactly.

Two Metallic Circuits Suspended on the Same Pole.[*] — If we have two circuits suspended on the same pole, the capacity of either circuit will be sensibly influenced by the proximity of the other circuit and the capacity will be larger than that given by formula (44). Suppose we have two circuits (1, 2) and (3, 4) on the same pole, as shown in Fig. 29. The capacity of circuit (1, 2) may be expressed by the formula

$$C = \frac{\log_e \frac{d_6}{a} \times 0.179}{4 \log_e \frac{d_5}{a} \log_e \frac{d_6}{a} - \left(\log_e \frac{d_3 d_2}{d_1 d_4}\right)^2} \text{ mf. per mile of loop conductor.} \quad (52)$$

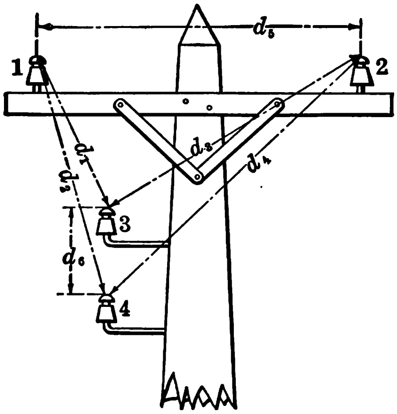

FIG. 29.

If circuit (3, 4) were removed to an infinite distance from circuit (1, 2) we should have $\frac{d_3 d_2}{d_1 d_4} = 1$ and $\log_e \frac{d_3 d_2}{d_1 d_4} = 0$.

Equation (52) would reduce to

$$C = \frac{0.179}{4 \log_e \frac{d_5}{a}} \text{ mf. per mile of loop conductor,}$$

which is formula (44).

If the circuits are separated by a distance of 5 feet or more, the capacity is modified but slightly on account of the proximity of the other circuit. If, however, the circuits are brought close together, the capacity may be affected materially.

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[*] Louis Cohen, *London Electrician*, Feb. 14, 1913.

Example. —

$a = 0.1$ in. $d = 2$ ft. $d_6 = 2$ ft. $d_1 = 1$ ft.
$d_2 = 3$ ft. $d_3 = 2.24$ ft. $d_4 = 3.6$ ft.

$$\log_e \frac{d_6}{a} = \log_e \frac{24}{0.1} = \log_e 240 = 5.480,$$

$$\log_e \frac{d_5}{a} = \log_e \frac{24}{0.1} = \log_e 240 = 5.480.$$

$$\log_e \frac{d_2 d_3}{d_1 d_4} = \log_e 1.866 = 0.624.$$

$$C = \frac{5.48 \times 0.179}{4 \times 5.48 \times 5.48 - (0.624)^2} = 0.00824 \text{ mf. per mile, loop wire.}$$

If circuit (3, 4) were removed, the capacity of circuit (1, 2) would be

$$C = \frac{0.179}{4 \times 5.48} = 0.00812 \text{ mf. per mile of loop wire.}$$

The presence of circuit (3, 4) has increased the capacity of (1, 2) by 1.5 per cent.

Three-phase Transmission Lines.* — If the spacings are symmetrical, as in the case when the wires are placed at the vertices of an equilateral triangle, and the voltages on the lines are balanced and equal, the capacity for each wire is the same as for a single-phase transmission line,

$$C = \frac{0.0894}{\log_e \frac{d}{a}} \text{ mf. per mile.}$$

The value of C as given above is the capacity between each wire and neutral, and

d = distance between wires,
a = radius of wire.

Charging current per wire is

$$I = \frac{E\omega C}{\sqrt{3} \times 10^6} \text{ amperes.} \qquad (53)$$

For unsymmetrical spacing, the calculations of the capacities or charging current in a three-phase line is very much involved, and is not of any great practical importance. It is therefore omitted here.

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* F. A. C. Perrine and F. G. Baum, *Trans. Am. Inst. E. E.*, May, 1900.

We may get an idea of the error introduced by using formula (44) for unsymmetrical spacings by determining the limiting values of the capacities as given by that formula.

As an illustration let us consider the case of the three wires arranged in a horizontal plane. The distance between wires is 12 feet, and the conductors being No. 0 B. & S., the radius $a = 0.163$ inches, taking $d = 12$ feet we have

$$\log_e \frac{d}{a} = 6.78,$$

$$C = \frac{0.0894}{6.78} = 0.0132 \text{ mf. per mile.}$$

For $d = 24$ ft., $\log_e \frac{d}{a} = 7.37,$

$$C = \frac{0.0894}{7.37} = 0.0121 \text{ mf. per mile.}$$

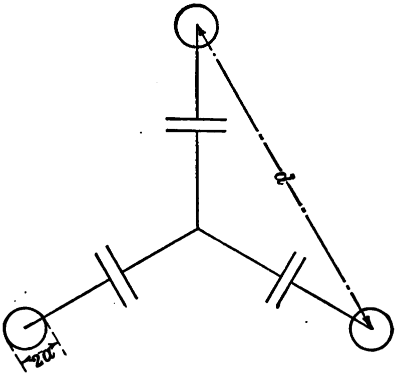

Fig. 30.

Prof. Karapetoff* states that Mr. J. G. Pertsch, Jr., found that, with certain simplifying assumptions, and when the wires are transposed, the equivalent spacing for the inductance and capacity is equal to the geometric mean of the three actual spacings, or

$$d_{eq} = \sqrt[3]{d_{12}\, d_{23}\, d_{13}}.$$

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* For a detailed discussion of the charging currents in a three-phase line with unsymmetrical spacings, see "The Electric Circuit" by V. Karapetoff, p. 199; and an article by G. S. Humphrey, *Electrical World*, Vol. 58, p. 1300, Nov. 25, 1911. See also "Inductance and Capacity of Three-phase Transmission Lines," by Louis Cohen, *Electrician*, July 18, 1913.

In the above example for instance,

$$d_{eq} = \sqrt[3]{12 \times 12 \times 24} = 15.12, \quad \log_e \frac{d_{eq}}{a} = \log_e \frac{15.12 \times 12}{0.163} = 7.012,$$

$$C = \frac{0.0894}{7.012} = 0.0128 \text{ mf. per mile.}$$

Capacity of Horizontal Antennæ.*—A horizontal antenna is generally made up of a number of wires in parallel suspended at a considerable height above ground. To obtain an expression for the capacity of such a network of conductors is rather a difficult matter. We may, however, obtain an approximate value of the capacity of antennæ by an indirect method.

For linear conductors we have the following relation between the inductance and capacity:

$$V^2 = \frac{1}{LC}, \qquad C = \frac{1}{LV^2}, \tag{54}$$

where L is the inductance in cms. per cm., C is the capacity in cms. per cm. expressed in electromagnetic units and V is the velocity of light ($V = 3 \times 10^{10}$ cms. per cm.). The relation given by equation (54) is only strictly true when there is no energy loss in the conductor, that is with perfect conductivity. The expression for C as given by equation (54) gives fairly accurate results if in the evaluation of L we neglect that part of the inductance which is due to the magnetic field within the conductor, and consider only the inductance which is due to the external magnetic field of the conductor.

In Chapter II we have given a set of formulæ, (65) to (71), for the calculations of the inductances of two, three, four and five wires in parallel, and we have also indicated a method for the approximate determination of the inductance of any number of wires in parallel; hence, knowing the inductance we can use formula (54) to determine the capacity.

As an illustration let us consider an antenna of the following constants:

a (radius of wire) = 0.08 in.,
d distance between adjacent wires = 2 feet,
h height above ground = 80 feet,
10 wires in parallel and 150 feet long.

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* Louis Cohen, *Electrician*, Feb. 21, 1913.

The self inductance of a single wire, neglecting the internal magnetic field of the conductor, is

$$L = 2\log_e \frac{2h}{a} = 2\log_e \frac{160 \times 12}{0.08} = 18.65.$$

The mutual inductance between two adjacent wires is

$$M = \log_e \frac{4h^2}{d^2} = \log_e \frac{4 \times 6400}{4} = 8.76.$$

By formula (70), Chapter II, we find the inductance of five wires in parallel,

$$\mathcal{L}' = 9.51 \text{ cms. per cm.}$$

The mutual inductance between two wires placed at a distance of 5 d is

$$M' = \log_e \frac{4 \times (80)^2}{10^2} = 5.54 \text{ cms.},$$

and the total inductance of the 10 wires in parallel is

$$\mathcal{L} = \frac{L' + M'}{2} = \frac{9.51 + 5.54}{2} = 7.52 \text{ cms. per cm.}$$

The capacity of the antenna is

$$C = \frac{10^{15}\,l}{9 \times 10^{20} \times 7.52} = \frac{10^{15} \times 150 \times 30.5}{9 \times 10^{20} \times 7.52} = 0.00068 \text{ mf.}$$

Concentric Cylinders.—The capacity of a condenser formed by two concentric cylinders is

$$\left.\begin{aligned} C &= \frac{Kl}{2\log_e \frac{r_2}{r_1}} \\ &= \frac{0.0894\,K}{\log_e \frac{r_2}{r_1}} \text{ mf. per mile,} \end{aligned}\right\} \quad (55)$$

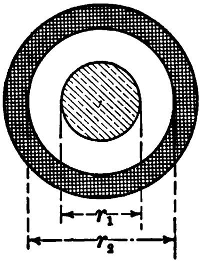

FIG. 31.

where

r_2 = inside radius of outer cylinder,
r_1 = radius of inner cylinder,
K = specific inductive capacity.

If the dielectric between the cylinders is not homogeneous, but consists of several cylindrical layers of different specific inductive capacities, formula (55) assumes the form

$$\left.\begin{aligned} C &= \frac{0.0894}{\sum_{i=1}^{i=n} \frac{1}{K_i} \log_e \frac{r_i}{r_{i-1}}} \\ &= \frac{0.0894}{\frac{1}{K_1}\log_e \frac{r_1}{r_0} + \frac{1}{K_2}\log_e \frac{r_2}{r_1} + \frac{1}{K_3}\log_e \frac{r_3}{r_2} + \cdots}. \end{aligned}\right\} \quad (56)$$

Example. —

r_1, radius of inner cylinder = 1 cm.,
r_2, inner radius of outer cylinder = 2 cm.

The dielectric consists of two layers each one-half cm. in radial thickness, the specific inductive capacities being 5 and 3, respectively.

Introducing these values in formula (56) we get

$$C = \frac{0.0894}{\frac{1}{5}\log_e \frac{1.5}{1} + \frac{1}{3}\log_e \frac{2}{1.5}} = \frac{0.0894}{0.0811 + 0.0959} = 0.505 \text{ mf. per mile.}$$

A uniform dielectric of specific inductive capacity 3 or 5 would give, in the above example, a capacity of 0.39 mf. per mile or 0.65 mf. per mile, respectively.

If the axes of the cylinders are displaced with respect to each other by a distance d, the capacity is given by

$$C = \frac{0.0894\,K}{\log_e (\beta + \sqrt{\beta^2 - 1})} \text{ mf. per mile,} \quad (57)$$

where $$\beta^2 = \frac{r_2^2 + r_1^2 - d^2}{2\,r_2 r_1}.$$

If d^2 is small compared with $r_2^2 - r_1^2$, then approximately,

$$C = \frac{0.0894\,K}{\log_e \left(\frac{r_2}{r_1} - \frac{r_2}{r_1}\frac{d^2}{r_2^2 - r_1^2}\right)} \text{ mf. per mile.} \quad (58)$$

Another approximate formula which has been derived for this case is as follows:

$$C = \frac{0.0894\,K}{\log_e \frac{r_2}{r_1}} \left\{1 + \frac{d^2}{(r_2^2 - r_1^2)\log_e \frac{r_2}{r_1}}\right\} \text{ mf. per mile.} \quad (59)$$

By comparing formulæ (58) and (59) with (55) it is obvious from inspection that the capacity is a minimum when the axes of the cylinders coincide. As an illustration of the relative accuracy of formulæ (58) and (59) the computed values of the capacities by formulæ (57), (58) and (59) for $\frac{r_2}{r_1} = 4$ and different values of d are given in the following table: ($K = 1$.)

r_2.	r_1.	d.	Formula 57.	Formula 58.	Formula 59.
4	1	0	0.0645	0.0645	0.0645
4	1	1	0.0676	0.0676	0.6750
4	1	2	0.0837	0.0829	0.0773
4	1	2.5	0.1111	0.1054	0.0837

The results by formulæ (57) and (58) agree very closely for all values of d, while formula (59) is applicable only when d is small. For comparatively large values of d, the error in formula (59) is considerable.

Capacities of Cables.* — In the discussion of capacities of cables it will be convenient to introduce the term "effective capacity," which we shall designate throughout by C_e. By the term effective capacity we mean that factor which, multiplied by the time rate of variation of electromotive force, will give the value of the charging current, that is, $I = C_e\, 2\, \pi n E$. In the case of a simple condenser, the effective capacity is merely the capacity of the condenser, but in the case of cables the effective capacity varies with any alteration in the other parts of the circuits as for instance when the lead sheath is grounded or insulated, or any of the other cores are grounded, etc.

In engineering problems we are mostly interested in knowing the values of the charging currents for a certain e.m.f. and a certain frequency, hence it will be very useful to obtain formulæ for the effective capacity of any cable system.

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* Alex. Russell, "Theory of Alternating Currents," Vol. I, Chapters IV and V.

L. Lichtenstein, "Capacity of Cables," Elektrotech. Zs., Vol. 25, pp. 106–110 and 124–126, Feb., 1904.

L. Lichtenstein, Beiträge zur Theorie der Kabel; Dissertation Königlichen Technischen Hochschule in Berlin, 1908.

Concentric Cable. — We shall consider first the case of a simple cable having a lead sheath and an iron sheath and insulating dielectrics between them.

Let

r_1 = the radius of conductor,
r_2 = the inner radius of lead sheath,
r_3 = the outer radius of lead sheath,
r_4 = the inner radius of iron sheath.

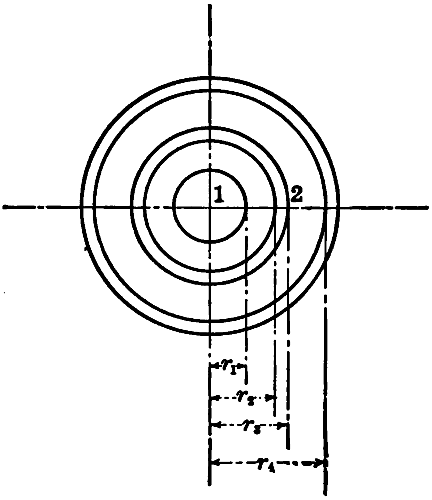

FIG. 32.

K_1 is the specific inductive capacity of insulating material between conductor and lead sheath and K_2 is the specific inductive capacity of insulating material between lead and iron sheath.

If the lead sheath is grounded the capacity of the cable is simply that of a cylindrical condenser formed by conductor and lead sheath, that is, by formula (55)

$$C = \frac{K_1 \, 0.0894}{\log_e \frac{r_2}{r_1}} \text{ mf. per mile.} \tag{60}$$

If, however, the lead sheath is insulated, charges are induced on the outer surface of the lead sheath and the inner surface of the iron sheath and thus another condenser is formed between lead and iron sheaths whose capacity is

$$C' = \frac{K_2\,0.0894}{\log_e \frac{r_4}{r_3}} \text{ mf. per mile.} \tag{61}$$

The effective capacity of the cable is that of the two condensers in series, that is,

$$C_e = \frac{CC'}{C + C'}. \tag{62}$$

If $K_1 = K_2$ we have

$$\left.\begin{aligned} C_e &= \frac{K\,0.0894}{\log_e \frac{r_2}{r_1} + \log_e \frac{r_4}{r_3}}, \\ &= \frac{K\,0.0894}{\log_e \frac{r_2 r_4}{r_1 r_3}} \text{ mf. per mile.} \end{aligned}\right\} \tag{63}$$

The capacity of the cable is reduced when the lead sheath is insulated.

Example. —

$$r_1 = 1 \text{ cm.,} \quad r_1 = 1.5 \text{ cm.,} \quad r_3 = 1.75 \text{ cm.,} \quad r_4 = 2.5 \text{ cm.,}$$

$$\log_e \frac{r_2}{r_1} = 0.405, \quad \log_e \frac{r_4}{r_3} = 0.356, \quad \log_e \frac{r_4 r_3}{r_1 r_2} = 0.761.$$

$$\begin{aligned} C &= 0.221\,K \text{ mf. per mile,} \\ C' &= 0.251\,K \text{ mf. per mile,} \\ C_e &= 0.118\,K \text{ mf. per mile.} \end{aligned}$$

In this case, by insulating the lead sheath, the capacity of the cable is reduced to about 50 per cent of its value when the lead sheath is grounded.

Triple Concentric Cable. —

Let r_1 = the radius of inner conductor,
r_2 = the inner radius of second conductor,
r_3 = the outer radius of second conductor,
r_4 = the inner radius of third conductor,
r_5 = the outer radius of third conductor,
r_6 = the inner radius of lead sheath.

We have in this case three cylindrical condensers formed between the several conductors, whose capacities are,

$$C_1 = \frac{K_1\,0.0894}{\log_e \frac{r_2}{r_1}} \text{ mf. per mile,}$$

$$C_2 = \frac{K_2\,0.0894}{\log_e \frac{r_4}{r_3}} \text{ mf. per mile,}$$

$$C_3 = \frac{K_3\,0.0894}{\log_e \frac{r_6}{r_5}} \text{ mf. per mile.}$$

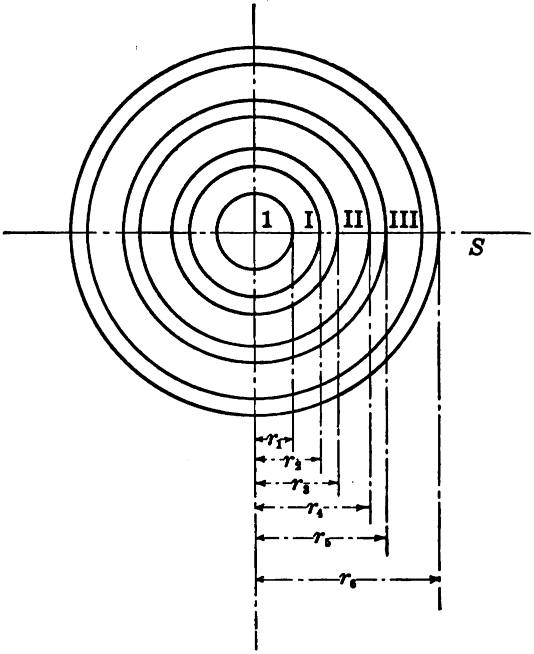

Fig. 33.

We shall assume that the three conductors are supplied from a three-phase generator delivering an electromotive force of pure sine form, the stator being star-connected.

If the neutral point is grounded, the charging currents have the values

$$\left.\begin{aligned}
I_1 &= \omega E\sqrt{3}\,C_1\sin\left(\omega t+\frac{\pi}{3}\right),\\
I_2 &= \omega E\left\{-C_1\sqrt{3}\sin\left(\omega t+\frac{\pi}{3}\right)-C_2\sqrt{3}\sin\omega t\right\},\\
I_3 &= \omega E\left\{C_2\sqrt{3}\sin\omega t-C_3\sin\left(\omega t+\frac{5\pi}{6}\right)\right\},\\
I_0 &= \omega EC_3\sin\left(\omega t+\frac{5\pi}{6}\right).
\end{aligned}\right\} \qquad (64)$$

The charging currents consist of two components differing in phase from each other and we cannot, therefore, speak of effective capacity in the case of a triple concentric cable. The effective values of the charging currents may be obtained graphically.

If the stator winding is insulated, neutral point not being grounded, we have for the charging currents

$$\left.\begin{aligned}
I_1 &= \omega EC_1\sqrt{3}\cos\left(\omega t-\frac{\pi}{6}\right),\\
I_2 &= \omega E\left\{-C_1\sqrt{3}\cos\left(\omega t-\frac{\pi}{6}\right)+C_2\sqrt{3}\sin\omega t\right\},\\
I_3 &= -\omega E\sqrt{3}\,C_2\sin\omega t.
\end{aligned}\right\} \qquad (65)$$

In this case also the charging currents can be best obtained graphically.

Capacity of Two-core Cable. — The effective capacity of a two-core cable can be expressed for all possible arrangements in terms of C_1 and C_{12}, where C_1 is the capacity of No. 1 conductor, when No. 2 and the sheath are grounded and C_{12} is the mutual capacity between the two conductors. The values of C_1 and C_{12} in terms of the dimensions of the cable are,

$$\left.\begin{aligned}
C_1 &= \frac{\log_e\left(\dfrac{R^2-d^2}{Rr}\right)\times 0.0894\,K}{\left(\log_e\dfrac{R^2-d^2}{Rr}\right)^2-\left(\log_e\dfrac{d^2+R^2}{2\,dR}\right)^2}\ \text{mf. per mile},\\
-C_{12} &= \frac{\log_e\dfrac{d^2+R^2}{2\,dR}\times 0.0894\,K}{\left(\log_e\dfrac{R^2-d^2}{Rr}\right)^2-\left(\log_e\dfrac{d^2+R^2}{2\,dR}\right)^2}\ \text{mf. per mile},
\end{aligned}\right\} \qquad (66)$$

where R is internal radius of sheath,
r is the radius of either conductor,
d is the distance between centers of sheath and either conductor.

It is assumed that both conductors are of the same radius and are at the same distance from center of sheath.

If the sheath is grounded and the two conductors are connected to a generator supplying an alternating e.m.f., we shall have for the charging current

$$I = C_e \omega E$$

and $$C_e \text{ (effective capacity)} = \frac{1}{2}(C_1 - C_{12}). \tag{67}$$

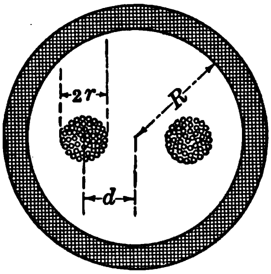

Fig. 34.

If we introduce the values of C_1 and C_{12} from (66) we shall have

$$C_e = \frac{K \times 0.0447}{\log_e\left(\frac{R^2 - d^2}{R^2 + d^2} \times \frac{2d}{r}\right)} \text{ mf. per mile.} \tag{68}$$

Let us suppose now that conductor 2 and the sheath are both grounded. In this case the capacity of No. 1 conductor is simply C_1 and that of No. 2 conductor, C_{12}; that is the charging currents on the two conductors and sheath are

$$\left.\begin{aligned} I_1 &= C_1 \omega E \cos \omega t, \\ I_2 &= C_{12} \omega E \cos \omega t, \\ I_s &= -(C_1 + C_{12}) \omega E \cos \omega t. \end{aligned}\right\} \tag{69}$$

If one terminal of generator is connected to the two conductors in parallel and the other terminal to the sheath which is grounded, the effective capacity of either conductor is

$$C_e = C_1 + C_{12}$$

and the total charging current on the two conductors is

$$I = 2(C_1 + C_{12}) \omega E \cos \omega t. \tag{70}$$

Evidently the charging current on sheath is

$$I_s = -I = -2(C_1 + C_{12}) \omega E \cos \omega t.$$

In all the above cases we assumed that the lead sheath was grounded. If the lead sheath is insulated, and one terminal of alternator is connected to the two conductors in parallel and the other to ground, we have for the effective capacity

$$C_e = \frac{CC_0}{C + C_0}, \tag{71}$$

where $$C_0 = 2\,(C_1 + C_{12})$$

and C is capacity of cylindrical condenser formed between lead and iron sheaths.

Example. —

$$r = 0.5 \text{ cm.}, \quad d = 1.0 \text{ cm.}, \quad R = 2.0 \text{ cm.}$$

$$C_1 = \frac{\log_e (4 - 1) \times 0.0894\,K}{\left(\log_e \frac{4-1}{1}\right)^2 - \left(\log_e \frac{1+4}{1}\right)^2} = 0.084\,K \text{ mf. per mile.}$$

$$-\,C_{12} = \frac{\log_e \frac{5}{4} \times 0.0894\,K}{\left(\log_e \frac{4-1}{1}\right)^2 - \left(\log_e \frac{5}{4}\right)^2} = 0.016\,K \text{ mf. per mile.}$$

For the case given by formula (68)

$$C_e = 0.051\,K \text{ mf. per mile.}$$

For the case given by formula (70)

$$C_e = 0.134\,K \text{ mf. per mile.}$$

For the case given by formula (71) we must also determine the value of C. We shall assume that the outer radius of lead sheath is 2.3 cm. and inner radius of iron sheath is 2.6 cm.; then,

$$C = \frac{0.0894\,K}{\log_e \frac{2.6}{2.3}} = 0.732\,K \text{ mf. per mile,}$$

$$2\,(C_1 + C_{12}) = 0.134\,K \text{ mf. per mile}$$

and $$C_e = \frac{CC_0}{C + C_0} = 0.114\,K \text{ mf. per mile.}$$

Capacity of Three-core Cables. — As in the case of a two-core cable we can express the capacities of the various possible arrangements in terms of C_1 and C_{12}. It is assumed that the conductors are symmetrically placed with respect to the axis of the sheath.

The formulæ for C_1 and C_{12} are

$$\left.\begin{aligned} C_1 &= \frac{\alpha_{11}{}^2 - \alpha_{12}{}^2}{\alpha_{11}{}^3 - 3\,\alpha_{11}\alpha_{12}{}^2 + 2\,\alpha_{12}{}^3}, \\ C_{12} &= \frac{\alpha_{12}{}^2 - \alpha_{11}\alpha_{12}}{\alpha_{12}{}^3 - 3\,\alpha_{11}\alpha_{12}{}^2 + 2\,\alpha_{12}{}^3}, \end{aligned}\right\} \qquad (72)$$

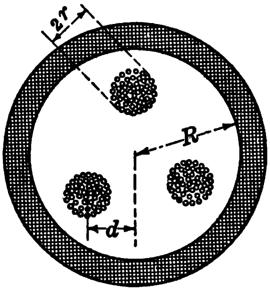

Fig. 35.

where

$$\alpha_{11} = 2\log_e \frac{R^2 - d^2}{Rr},$$

$$\alpha_{12} = \log_e \left\{ \frac{1}{3}\left(\frac{R^2}{d^2} + 1 + \frac{d^2}{R^2} \right) \right\},$$

R = radius of sheath,
r = radius of conductor,
d = distance between axes of conductor and axis of sheath.

An equivalent formula to (72) is the following:

$$\left.\begin{aligned} C_{12} &= \frac{1}{6\log_e \dfrac{R^6 - d^6}{3\,R^3 d^2 r}} - \frac{1}{6\log_e \left\{ \dfrac{d\sqrt{3}}{r} \times \dfrac{R^2 - d^2}{(R^4 + R^2 d^2 + d^4)^{\frac{1}{2}}} \right\}} \\ C_1 &= \frac{1}{6\log_e \dfrac{R^6 - d^6}{3\,R^3 d^2 r}} + \frac{1}{3\log_e \left\{ \dfrac{d\sqrt{3}}{r} \times \dfrac{R^2 - d^2}{(R^4 + R^2 d^2 + d^4)^{\frac{1}{2}}} \right\}} \end{aligned}\right\} \qquad (73)$$

Formulæ (72) and (73) were both derived by the aid of the principles of images and give identical results.

The values of C_1 and C_{12} in formulæ (72) and (73) are given in electrostatic units. To convert them to microfarads per mile we must multiply by the numerical factor 0.179 and, since the conductors are embedded in a dielectric, we must multiply by K the specific inductive capacity of dielectric. As an illustration we shall consider a cable of the following dimensions:

$$r = 0.5 \text{ cm.}, \quad R = 2.5 \text{ cm.}, \quad d = 1.3 \text{ cm.}$$

$$\alpha_1 = 2 \log_e 3.65 = 2.6,$$

$$\alpha_{12} = \log_e 1.66 = 0.51,$$

$$C_1 = \frac{\{(2.6)^2 - (0.51)^2\}\, 0.179\, K}{(2.6)^3 - 3 \times 2.6 \times (0.51)^2 + 2\,(0.51)^3} = 0.074\, K \text{ mf. per mile},$$

$$C_{12} = \frac{\{(0.51)^2 - 2.6 \times 0.51\}\, K \times 0.179}{(2.6)^3 - 3 \times 2.6 \times (0.51)^2 + 2(0.51)^2} = -0.0119 \text{ mf. per mile.}$$

By formula (73) we find for the same example,

$$\frac{R^2 - d^2}{(R^4 + R^2d^2 + d^4)^{\frac{1}{2}}} = 0.63,$$

$$\frac{R^6 - d^6}{3\,R^3d^2r} = 6.04, \qquad \frac{d\sqrt{3}}{2} = 4.5,$$

$$\log_e \frac{R^6 - d^6}{3\,R^3d^2r} = 1.8, \qquad \log_e \left\{ \frac{d\sqrt{3}}{2} \times \frac{R^2 - d^2}{(R^4 + R^2d^2 + d^4)^{\frac{1}{2}}} \right\} = 1.04$$

and

$$C_1 = 0.074\, K \text{ mf. per mile},$$
$$C_{12} = -\ 0.0119\, K \text{ mf. per mile.}$$

From this example it is seen that both formulæ give identical results.

We shall assume now that the three cores of the cable are connected to a three-phase star-connected generator supplying an e.m.f. of pure sine form, the neutral point and the lead sheath being grounded. In this case the effective capacity of each conductor is

$$C_e = C_1 - C_{12}. \tag{74}$$

The charging currents on the three conductors are equal to each other in amplitude and differ, of course, in phase by 120 degrees; that is,

$$\left.\begin{aligned} I_1 &= \omega C_e E_m \cos \omega t, \\ I_2 &= \omega C_e E_m \cos\left(\omega t - \frac{2\pi}{3}\right), \\ I_3 &= \omega C_e E_m \cos\left(\omega t - \frac{4\pi}{3}\right). \end{aligned}\right\} \tag{75}$$

If we ground one conductor and connect the other two conductors to a single phase generator, the effective capacity of each conductor is

$$C_e = \frac{C_1 - C_{12}}{2}. \tag{76}$$

The charging currents on the two conductors are equal and opposite in sign;

$$I_1 = -I_2 = C_e \omega E_m \cos \omega t.$$

A three-core cable is supplied with a single phase alternating current, two conductors connected in parallel for the transmission of the outgoing current and the third conductor being the return, the lead sheath grounded. Such an arrangement offers the somewhat peculiar condition that for the same e.m.f. the charging currents on the conductors differ when the neutral point of the winding is grounded or insulated. That is to say, the effective capacities of the conductor depend on the electrical condition of the stator winding, grounded or insulated.

In the first case when the neutral point of stator winding is grounded the effective capacities of the three conductors, 1, 2 and 3, are $\frac{1}{2} C_1$, $\frac{1}{2} C_1$, $\frac{1}{2} (C_1 - 2 C_{12})$, respectively.

The charging currents are

$$\left.\begin{aligned} I_1 = I_2 &= \tfrac{1}{2} C_1 \omega E_m \cos \omega t, \\ I_3 &= -\tfrac{1}{2} (C_1 - 2 C_{12}) \omega E_m \cos \omega t. \end{aligned}\right\} \tag{77}$$

The charging current on the inner surface of the sheath is

$$I_s = -\tfrac{1}{2} (C_1 + 2 C_{12}) \omega E_m \cos \omega t,$$

and

$$I_1 + I_2 + I_3 + I_s = 0.$$

It is evident from (77) that the currents in the outgoing and return conductors are not equal.

When the stator winding is insulated, the effective capacities of the three conductors, 1, 2 and 3, are

$$\tfrac{1}{3} (C_1 - C_{12}), \tfrac{1}{3} (C_1 - C_{12}) \text{ and } \tfrac{2}{3} (C_{12} - C_1), \text{ respectively.}$$

The charging currents are

$$\left.\begin{aligned} I_1 = I_2 &= \tfrac{1}{3} (C_1 - C_{12}) \omega E_m \cos \omega t, \\ I_3 &= \tfrac{2}{3} (C_{12} - C_1) \omega E_m \cos \omega t. \end{aligned}\right\} \tag{78}$$

Evidently, as required,

$$I_1 + I_2 + I_3 = 0.$$

The above example shows that the effective capacity depends not only on the arrangement of the conductors but also on the electrical condition of the stator winding with respect to the cable, that is grounded or insulated.

A list of the capacities that can be obtained from a three-core cable is given in Russell's "Alternating Current Theory" as follows:

(1) Capacity between 1 and 2 (3 grounded) $= \frac{1}{2}(C_1 - C_{12})$.

(2) Capacity between 1 and 2, 3 $= \frac{2}{3}(C_1 - C_{12})$.

(3) Capacity between 1 and S (2 and 3 insulated)

$$= \frac{(C_1 - C_{12})\,(C_1 + 2\,C_{12})}{C_1 + C_{12}}.$$

(4) Capacity between 1 and S, 2 (3 insulated)

$$= \frac{(C_1 - C_{12})\,(C_1 + C_{12})}{C_1}.$$

(5) Capacity between 1 and S, 2, 3 $= C_1$.

(6) Capacity between S and 1, 2 (3 insulated)

$$= \frac{2\,(C_1 - C_{12})\,(C_1 + 2\,C_{12})}{C_1}.$$

(7) Capacity between 1, S and 2, 3 $= 2\,(C_1 + C_{12})$.

(8) Capacity between S and 1, 2, 3 $= 3\,(C_1 + 2\,C_{12})$.

Example.—Taking the values of C_1 and C_{12} as given on page 119,

$$C_1 = 0.074 \text{ mf. per mile,}$$
$$C_{12} = -\,0.0119 \text{ mf. per mile,}$$

we get

(1) Capacity between 1 and 2 = 0.043 mf. per mile.

(2) Capacity between 1 and 2, 3 = 0.0573 mf. per mile.

(3) Capacity between 1 and S (2 and 3 insulated) = 0.0695 mf. per mile.

(4) Capacity between 1 and S, 2 (3 insulated) = 0.072 mf. per mile.

(5) Capacity between 1 and S, 2, 3 = 0.074 mf. per mile.

(6) Capacity between S and 1, 2 (3 insulated) = 0.1165 mf. per mile.

(7) Capacity between 1, S and 2, 3 = 0.1242 mf. per mile.

(8) Capacity between S and 1, 2, 3 = 0.1863 mf. per mile.

TABLE XI

Values of C and C'

For use in Formulæ (25) and (26)

$y = \frac{d}{a}$.	$\frac{C}{a}$.	$\frac{C'}{a}$.	$y = \frac{d}{a}$.	$\frac{C}{a}$.	$\frac{C'}{a}$.
2.0_61	4.6647	0.6931	2.6	0.8552	0.7340
2.0_51	4.0891	0.6931	2.7	0.8275	0.7401
2.0_41	3.5134	0.6931	2.8	0.8044	0.7460
2.0_31	2.9378	0.6931	2.9	0.7847	0.7517
2.0_21	2.3626	0.6932	3.0	0.7677	0.7572
2.01	1.7896	0.6942	3.1	0.7528	0.7625
2.02	1.6191	0.6946	3.2	0.7396	0.7677
2.03	1.5202	0.6954	3.3	0.7278	0.7726
2.04	1.4506	0.6961	3.4	0.7172	0.7774
2.05	1.3970	0.6968	3.5	0.7076	0.7820
2.06	1.3536	0.6975	3.6	0.6989	0.7864
2.07	1.3171	0.6983	3.7	0.6909	0.7907
2.08	1.2857	0.6990	3.8	0.6836	0.7948
2.09	1.2582	0.6997	3.9	0.6768	0.7988
2.10	1.2337	0.7004	4.0	0.6705	0.8026
2.15	1.1413	0.7040	5.0	0.6263	0.8345
2.20	1.0775	0.7075	6.0	0.6006	0.8577
2.25	1.0291	0.7117	7.0	0.5836	0.8753
2.30	0.9911	0.7144	8.0	0.5712	0.8891
2.35	0.9594	0.7178	9.0	0.5626	0.9001
2.40	0.9326	0.7211	10.0	0.5556	0.9092
2.45	0.9095	0.7245	100	0.5051	0.9901
2.50	0.8892	0.7277	1000	0.5005	0.9990

TABLE XII

$\frac{d}{2a}$.	$\log_e \frac{d}{a}$.	$\cosh^{-1} \frac{d}{2a}$.	$\frac{d}{2a}$.	$\log_e \frac{d}{a}$.	$\cosh^{-1} \frac{d}{2a}$.
1.01	0.7031	0.1413	3.5	1.9459	1.9248
1.05	0.7419	0.3149	4.0	2.0794	2.0634
1.1	0.7885	0.4435	4.5	2.1972	2.1846
1.2	0.8755	0.6224	5.0	2.3025	2.2924
1.3	0.9555	0.7564	5.5	2.3979	2.3895
1.4	1.0296	0.8670	6.0	2.4849	2.4779
1.5	1.0986	0.9622	7.0	2.6390	2.6339
1.6	1.1631	1.0470	8.0	2.7726	2.7687
1.7	1.2238	1.1232	9.0	2.8903	2.8873
1.8	1.2809	1.1929	10.0	2.9956	2.9932
1.9	1.3350	1.2569	12	3.1780	3.1763
2.0	1.3863	1.3170	14	3.3322	3.3309
2.2	1.4816	1.4255	16	3.4657	3.4648
2.4	1.5686	1.5216	18	3.5835	3.5827
2.6	1.6487	1.6096	20	3.6888	3.6882
2.8	1.7228	1.6886	25	3.9119	3.9116
3.0	1.7918	1.7627			

TABLE XIII

Specific Inductive Capacities of Solids (air = unity)

Substance.	Specific inductive capacity.	Authority.
Calcspar parallel to axis	7.5	Romich and Nowak.
Calcspar perpendicular to axis	7.7	Romich and Nowak.
Caoutchouc	2.12–2.34	Schiller.
Caoutchouc, vulcanized	2.69–2.94	Schiller.
Celluvert, hard gray	1.19	Elsas.
Celluvert, hard red	1.44	Elsas.
Celluvert, hard black	1.89	Elsas.
Celluvert, soft red	2.66	Elsas.
Ebonite	2.08	Rossetti.
Ebonite	3.15–3.48	Boltzmann.
Ebonite	2.21–2.76	Schiller.
Ebonite	2.72	Winkelmann.
Ebonite	2.56	Wüllner.
Ebonite	2.86	Elsas.
Ebonite	1.9	Thomson (from Hertz's vibrations).
Fluor-spar	6.7	Romich and Nowak.
Fluor-spar	6.8	Curie.
Glass,* density 2.5 to 4.5	5–10	Various.
Double extra dense flint, density 4.5	9.90	Hopkinson.
Dense flint, density 3.66	7.38	Hopkinson.
Light flint, density 3.20	6.70	Hopkinson.
Very light flint, density 2.87	6.61	Hopkinson.
Hard carbon, density 2.485	6.96	Hopkinson.
Plate	8.45	Hopkinson.
Mirror	5.8–6.34	Schiller.
Mirror	6.46–7.57	Winkelmann.
Mirror	6.88	Doule.
Mirror	6.44–7.46	Elsas.

* The values here quoted apply when the duration of charge lies between 0.25 and 0.00005 of a second. J. J. Thomson has obtained the value 2.7 when the duration of the charge is about $\frac{1}{25 \times 10_6}$ of a second; and this is confirmed by Bloudlot, who obtained for a similar duration 2.8.

TABLE XIII (*Continued*)

SPECIFIC INDUCTIVE CAPACITIES OF SOLIDS (AIR = UNITY)

Substance.	Specific inductive capacities.	Authority.
Guttapercha	3.3–4.9	Submarine cable data.
Gypsum	6.33	Curie.
Mica	6.64	Klemenčič.
Mica	8.00	Curie.
Mica	7.98	Bouty.
Mica	5.66–597	Elsas.
Mica	4.6	Romich and Nowak.
Paraffin	2.32	Boltzmann.
Paraffin	1.98	Gibson and Barclay.
Paraffin	2.29	Hopkinson.
Paraffin, quickly cooled, translucent	1.68–1.92	Schiller.*
Paraffin, slowly cooled, white	1.85–2.47	Schiller.
Paraffin fluid, pasty	1.98–2.08	Arons and Rubens.
Paraffin, solid	1.95	Arons and Rubens.
Porcelain	4.38	Curie.
Quartz along the optic axis	4.55	Curie.
Quartz, transverse	4.49	Curie.
Resin	2.48–2.57	Boltzmann.
Rock salt	18	Hopkinson.
Rock salt	5.85	Curie.
Selenium	10.2	Romich and Nowak.
Shellac	3.10	Winkelmann.
Shellac	3.67	Doule.
Shellac	2.95–3.73	Wüllner.
Spermaceti	2.18	Rosetti.
Spermaceti	2.25	Felici.
Sulphur	3.84–3.90	Boltzmann.
Sulphur	2.24	J. J. Thomson.
Sulphur	2.94	Bloudlot.

* The lower values were obtained by electric oscillations of duration of charge about 0.0006 second. The larger values were obtained when duration of charge was about 0.02 second.

TABLE XIV

SPECIFIC INDUCTIVE CAPACITIES OF LIQUIDS

Substance.	Specific inductive capacity.	Authority.
Alcohols:		
Amyl	15–15.9	Cohen and Arons.
Ethyl	24–27	Various.
Methyl	32.65	Tereschin.
Propyl	22.8	Tereschin.
Anilin	7.5	Tereschin.
Benzene	1.93–2.45	Various.
Hexane, between 11° and 13° C.	1.859	Landolt and Jahn.
Octane, between 13.5° and 14°C.	1.934	Landolt and Jahn.
Decane, between 13.5° and 14.2° C.	1.966	Landolt and Jahn.
Amylene, between 15° and 16.2° C.	2.201	Landolt and Jahn.
Octylene, between 11.5° and 13.6° C.	2.175	Landolt and Jahn.
Decylene, between 16.7° C.	2.236	Landolt and Jahn.
Oils:		
Arachid	3.17	Hopkinson.
Castor	4.6–4.8	Various.
Colza	3.07–3.14	Hopkinson.
Lemon	2.25	Tomaszewski.
Neatsfoot	3.07	Hopkinson.
Olive	3.08–3.16	Arons and Rubens; Hopkinson.
Petroleum	2.02–2.19	Various.
Petroleum ether	1.92	Hopkinson.
Rape seed	2.2–3.0	Various.
Seasame	3.17	Hopkinson.
Sperm	3.02–3.09	Hopkinson.
Turpentine	2.15–2.28	Various.
Vaseline	2.17	Fuchs.
Ozokerite	2.13	Hopkinson.
Toluene	2.2–2.4	Various.
Xylene	2.3–2.6	Various.

CHAPTER IV

ALTERNATING CURRENT CIRCUITS

In alternating current circuits the distribution of the currents and potentials in any arrangement of circuits is governed by the frequency of the impressed electromotive force, the resistances, self and mutual inductances and capacities of the various branches of the circuit. A variation in any of the electrical constants may result in a change in the currents, in magnitude as well as in phase, in all the branches of the circuit. Every alternating current problem must therefore be considered separately and careful account must be taken of all the electrical constants of the circuit. Since there is a wide range of possible arrangements of circuits, we shall require a considerable number of formulæ to cover all the more important cases which may arise in practice.

It is also to be noted that the permanent value of the current, which in the case of alternating currents is a constant periodic function of the time, is reached only after an appreciable time interval when the circuit is closed. At the moment of closing or opening, or on any change in the electrical condition of circuits, a transient phenomena occurs which is usually active only for a very short interval. The formulæ for transient phenomena will be considered in Chapter V, while in this chapter we shall give a collection of formulæ for the permanent values of current and potential distribution in various arrangements of circuits.

In this and the succeeding chapters the following notation will be used:

Notation. —

r = resistance.
L = inductance.
C = capacity.
N = frequency.
$\omega = 2\pi N$.
$x_m = \omega L$ = inductive reactance.

$x_c = \dfrac{1}{\omega C}$ = capacity reactance.

$x = x_m - x_c$ reactance of circuit having inductance and capacity.

$Z = r + jx$ = impedance.

$z = \sqrt{r^2 + x^2}$ = absolute value of impedance.

$Y = \dfrac{1}{Z}$ = admittance.

$y = \sqrt{g^2 + b^2}$ = absolute value of admittance.

g = conductance.

b = susceptance.

$$g - jb = \frac{1}{r + jx} = \frac{r - jx}{r^2 + x^2}.$$

$g = \dfrac{r}{r^2 + x^2} = \dfrac{r}{z^2}$ = effective conductance in mhos, energy component of current.

$b = \dfrac{x}{r^2 + x^2} = \dfrac{x}{z^2}$ = effective susceptance in mhos, wattless component of current.

$r = \dfrac{g}{g^2 + b^2} = \dfrac{g}{y^2}$ = effective resistance in ohms, energy component of e.m.f.

$x = \dfrac{b}{g^2 + b^2} = \dfrac{b}{y^2}$ = effective reactance in ohms, wattless component of e.m.f.

Any alternating current wave can be resolved by the aid of Fourier's theorem into a series of waves, each one being a sine function in its variation with respect to time; hence in the discussion of alternating current problems it is nearly always justifiable to assume that the impressed electromotive force is of sine form. If it is a complex wave it can be resolved into its several harmonic components, and each one treated as a separate source of e.m.f.

If E is a periodic function with respect to time we can put it in the following form:

$$\begin{aligned} E = {} & A_1 \sin(\beta t - \phi_1) + A_2 \sin(2\beta t - \phi_2) + A_3 \sin(3\beta t - \phi_3) + \cdots \\ & + B_1 \cos(\beta t - \psi_1) + B_2 \cos(2\beta t - \psi_2) + B_3 \cos(3\beta t - \psi_3) + \cdots \quad (1) \end{aligned}$$

The alternating electromotive force and current curves which occur in practice have only odd harmonics. A complete discussion of the theory of harmonic analysis is beyond the scope of this book, and to give only an outline of the subject would not be of any practical value. The reader will find a good discussion on the general subject of harmonic analysis in Byerly's Fourier's Theory and Spherical Harmonics, and its application to the analysis of alternating current curves is quite fully discussed in the "Theorie der Wechselströme," by J. L. La Cour and O. S. Bragstad.

In alternating currents we have to distinguish between the maximum value, the effective value, which is the square root of the mean square, and the mean or average value. In the case of a simple harmonic function the effective value is $\frac{A}{\sqrt{2}}$ where A is the maximum value, and the mean or average for a half period is $\frac{2A}{\pi}$. The arithmetic mean for a whole period is zero. In Table XV we give the relation between effective, mean and maximum values for various shapes of curves.

TABLE XV

CHARACTERISTIC FEATURES OF DIFFERENT FORMS OF ALTERNATING CURRENT AND PRESSURE CURVES

Name of curve.	Ratio of average to maximum value.	Ratio of effective to maximum value.	Ratio of maximum to effective value.	Ratio of effective to average value.	Area of squared curve corresponding to unit maximum value.
Sinusoid	0.637	0.707	1.414	1.112	0.500
Semicircle	0.785	0.835	1.198	1.063	0.697
Parabolic curve	0.666	0.730	1.369	1.096	0.533
Triangle	0.500	0.577	1.732	1.155	0.333
Approximate rectangle	0.856	0.889	1.124	1.038	0.791
Rectangle	1.000	1.000	1.000	1.000	1.000

In the case of complex waves consisting of several harmonies the effective value is *

$$E_{\text{eff}} = \sqrt{\tfrac{1}{2}\left(E_1^2 + E_3^2 + E_5^2 + \cdots\right)}, \qquad (2)$$

where E_1, E_3, E_5, etc., are the maximum values of the several harmonic components.

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* See "Theorie der Wechselströme," von J. L. La Cour und O. S. Bragstad, Ch. XII.

Form factor. — The ratio of the effective value of a periodic curve to its mean value is called the form factor, because it varies with the form of the curve, the more peaked the curve the greater the value of the form factor. For a pressure curve the form factor is given by the expression

$$f = \frac{\sqrt{\frac{2}{T}\int_0^{\frac{T}{2}} E^2\,dt}}{\frac{2}{T}\int_0^{\frac{T}{2}} E\,dt}. \tag{3}$$

For a sine curve the form factor is

$$f = \frac{1}{\sqrt{2}} \div \frac{2}{\pi} = \frac{\pi}{2\sqrt{2}} = 1.112.$$

Curve factor. — The ratio of the effective value to the maximum value of the fundamental harmonic is called the curve factor.

$$\sigma = \frac{E}{E_1} = \sqrt{1 + \left(\frac{E_3}{E_1}\right)^2 + \left(\frac{E_5}{E_1}\right)^2 + \cdots} \tag{4}$$

As an illustration consider the case of an e.m.f. curve having three upper odd harmonics. The maximum values of the fundamental and the harmonics are

$$E_1 = 100, \quad E_3 = 40, \quad E_5 = 20, \quad E_7 = 10.$$

The effective value is

$$E_{\text{eff}} = \sqrt{(100)^2 + (40)^2 + (20)^2 + (10)^2} = 110.$$

The curve factor is

$$\sigma = \frac{E_{\text{eff}}}{E_1} = \frac{110}{100} = 1.1.$$

Alternating Current Circuits. — When an electromotive force of sinusoidal form, $E \cos \omega t$, is impressed on a circuit containing inductance and resistance, it generates a current in the circuit.

$$\left.\begin{aligned} I &= \frac{E}{\sqrt{x_m{}^2 + r^2}} \cos(\omega t - \phi), \\ \tan\phi &= \frac{x_m}{r}, \end{aligned}\right\} \tag{5}$$

the current lagging behind the electromotive force by the angle ϕ.

Example. —

$$r = 5 \text{ ohms}, \quad L = 0.01 \text{ henry}, \quad \omega = 2\pi \times 60, \quad E = 500 \text{ volts}.$$

$$x_m = \omega L = 3.73.$$

$$I_m = \frac{500}{\sqrt{(3.77)^2 + (5)^2}} = 80 \text{ amp.}$$

$$\tan\phi = \frac{3.77}{5} = 0.754, \quad \phi = 37^\circ\, 3'.$$

The maximum value of the current is 80 amperes and it lags 37° 3′ behind the e.m.f.

If the circuit contains capacity reactance and resistance, the current is

$$\left.\begin{aligned} I &= \frac{E}{\sqrt{x_c^2 + r^2}} \cos(\omega t + \psi), \\ \tan\psi &= \frac{x_c}{r}, \end{aligned}\right\} \tag{6}$$

the current leading the electromotive force by the angle ψ. The voltage on the condenser is

$$e = \frac{x_c E}{\sqrt{x_c^2 + r^2}} \sin(\omega t + \psi). \tag{7}$$

Example. —

$$r = 5 \text{ ohms}, \quad C = 0.002 \text{ farads}, \quad \omega = 2\pi \times 60, \quad E = 500 \text{ volts}.$$

$$x_c = \frac{1}{\omega C} = 1.324,$$

$$I_m = \frac{E}{\sqrt{x_c^2 + r^2}} = \frac{500}{\sqrt{(1.324)^2 + (5)^2}} = 96.7 \text{ amp.},$$

$$\tan\psi = \frac{1.324}{5} = 0.265, \quad \psi = 14^\circ\, 51'.$$

The current leads the e.m.f. by the angle 14° 51′. The voltage across the condenser is

$$e = \frac{1.324 \times 500}{\sqrt{(1.324)^2 + (5)^2}} = 128 \text{ volts.}$$

When the circuit contains both capacity reactance and inductive reactance we have

$$\left.\begin{aligned} I &= \frac{E}{\sqrt{(x_m - x_c)^2 + r^2}} \cos(\omega t - \gamma), \\ \tan\gamma &= \frac{x_m - x_c}{r}, \end{aligned}\right\} \tag{8}$$

When $x_m = x_c$, equation (8) reduces to

$$I = \frac{E}{r}\cos\omega t. \tag{9}$$

The circuit acts as if there is no reactance, and it is said to be in resonance with the frequency of the impressed electromotive force. The condition for resonance then is

$$\left.\begin{aligned} &x_m = x_c, \\ \text{or}\quad &\omega L = \frac{1}{\omega C}, \quad \omega^2 LC = 1, \\ \text{or}\quad &\omega = \frac{1}{\sqrt{LC}}. \end{aligned}\right\} \tag{10}$$

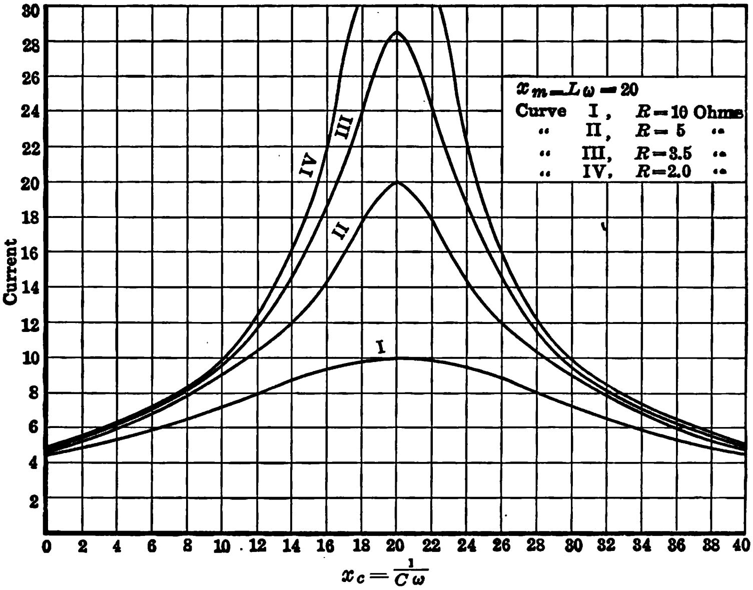

FIG. 36.—Resonance curves for different values of resistance in the circuit.

In any circuit containing inductance and capacity it is always possible to adjust those constants so as to produce resonance condition for a given frequency. For a given electromotive force, the current rises as the resonance condition is approached, which is more pronounced the smaller the resistance in the circuit. In Fig. 36 are plotted a series of resonance curves for different resistances.

The voltage across the condenser is

$$e = \frac{x_c E \sin(\omega t - \gamma)}{\sqrt{(x_m - x_c)^2 + r^2}}. \tag{11}$$

For resonance condition and small resistance, the voltage across the condenser may exceed many times the voltage of the generator.

Example. —

$$\omega = 2\pi \times 60 \quad C = 20 \text{ mf.}$$

$$x_c = \frac{1}{2\pi \times 60 \times 20 \times 10^{-6}} = 1325 = x_m, \; r = 20 \text{ ohms}, \; E = 100\, V.$$

$$e = \frac{1325}{20} E \sin \omega t = 66.2\, E \sin \omega t.$$

The voltage across condenser is about 6600 volts, 66 times the electromotive force of the generator.

If the electromotive force is a complex wave, that is, of the form

$$E = E_1 \cos(\omega t + \alpha_1) + E_3 \cos(3\,\omega t + \alpha_3) + E_5 \cos(5\,\omega t + \alpha_5) + \cdots$$

the current generated in the circuit is

$$I = \frac{E_1 \cos(\omega t + \alpha_1 - \gamma_1)}{\sqrt{(x_m - x_c)^2 + r^2}} + \frac{E_3 \cos(3\,\omega t + \alpha_3 - \gamma_3)}{\sqrt{\left(3\,x_m - \frac{x_c}{3}\right)^2 + r^2}}$$

$$+ \frac{E_5 \cos(5\,\omega t + \alpha_5 - \gamma_5)}{\sqrt{\left(5\,x_m - \frac{x_c}{5}\right)^2 1 + r^2}} + \cdots.$$

$$\tan \gamma_1 = \frac{x_m - x_c}{r}, \; \tan \gamma_3 = \frac{3\,x_m - \frac{x_c}{3}}{r}, \; \tan \gamma_5 = \frac{5\,x_m - \frac{x_c}{5}}{r} \cdots \tag{12}$$

It may happen that the inductance and capacity of the circuit are of such magnitudes as to bring the circuit into resonance for some harmonic of the wave, for instance we may have $3\,x_m - \frac{x_c}{3} = 0$, that is, the circuit is in resonance for the third harmonic. In that case the current produced by the third harmonic in the electromotive force wave may be larger than that produced by the fundamental, though the amplitude of the e.m.f. in the fundamental may be much larger than that of the third harmonic.

Example. —

$$C = 7.77 \text{ mf.}$$

$$L = 0.1 \text{ henry.}$$

$$\omega = 2\pi \times 60.$$

$$E_1 = 100, \quad E_3 = 10, \quad r = 10 \text{ ohms.}$$

$$x_m - x_c = 304, \quad 3\,x_m - \frac{x_c}{3} = 0.$$

$$\frac{E_1}{\sqrt{(x_m - x_c)^2 + r^2}} = \frac{100}{\sqrt{(304)^2 + (10)^2}} = 0.33 \text{ amp.}$$

$$\frac{E_3}{\sqrt{\left(3\,x_m - \frac{x_c}{3}\right)^2 + r^2}} = \frac{10}{10} = 1 \text{ amp.}$$

The current due to the first harmonic of the e.m.f. curve is three times as large as the current due to the fundamental, though the maximum value of the fundamental in the e.m.f. curve is ten times as large as the first harmonic.

If there are several impedances z_1, z_2, z_3 . . . connected in series in the circuit, the resultant is the algebraic sum of the separate impedances. Denoting by z_t the total impedance, we have

$$z_t = z_1 + z_2 + z_3 + \cdots = (r_1 + r_2 + r_3 + \cdots) + j\,(x_1 + x_2 + x_3 + \ldots), \tag{13}$$

and the absolute value of the current,

$$I = \frac{E}{\sqrt{(x_1 + x_2 + x_3 + \cdots)^2 + (r_1 + r_2 + r_3 + \cdots)^2}}. \tag{14}$$

The e.m.f. E_n across any impedance coil, say the nth coil, is

$$E_n = \frac{\sqrt{x_n{}^2 + r_n{}^2}}{\sqrt{(x_1 + x_2 + x_3 + \cdots)^2 + (r_1 + r_2 + r_3 + \cdots)^2}}. \tag{15}$$

Parallel Circuits.* — Two circuits in parallel, each having resistance and inductance. Denoting the reactances and impedances of the two branches by x_1, x_2 and z_1, z_2 respectively, the currents in the two branched circuits and the main circuit are

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* Alex. Russell, "Theory of Alternating Currents," Vol. I, Ch. VII. Andrew Gray, "Absolute Measurements in Electricity and Magnetism," Vol. II, Pt. I, Ch. IV.

$$I_1 = \frac{E}{z_1}\cos(\omega t - \phi_1),$$

$$I_2 = \frac{E}{z_2}\cos(\omega t - \phi_2),$$

$$I_0 = \frac{E}{\sqrt{\dfrac{1}{\left(\dfrac{r_1}{z_1^2} + \dfrac{r_2}{z_2^2}\right)^2 + \left(\dfrac{x_1}{z_1^2} + \dfrac{x_2}{z_2^2}\right)^2}}}\cos(\omega t - \phi_0),$$

$$\tan\phi_1 = \frac{x_1}{r_1}, \quad \tan\phi_2 = \frac{x_2}{r_2}, \quad \tan\phi_0 = \frac{\dfrac{x_1}{z_1} + \dfrac{x_2}{z_2}}{\dfrac{r_1}{z_1} + \dfrac{r_2}{z_2}} = \frac{x_1 z_2 + x_2 z_1}{r_1 z_2 + r_2 z_1}, \qquad (16)$$

or we may write

$$I_0 = \frac{E\cos(\omega t - \phi_0)}{\sqrt{r_0^2 + x_0^2}} = \frac{E}{z_0}\cos(\omega t - \phi_0),$$

where

$$r_0 = \frac{\dfrac{r_1}{z_1^2} + \dfrac{r_2}{z_2^2}}{\left(\dfrac{r_1}{z_1^2} + \dfrac{r_2}{z_2^2}\right)^2 + \left(\dfrac{x_1}{z_1^2} + \dfrac{x_2}{z_2^2}\right)^2}$$ is the equivalent resistance of the circuit and

$$x_0 = \frac{x_1}{z_1^2} + \frac{x_2}{z_2^2} \div \left(\frac{r_1}{z_1^2} + \frac{r_2}{z_2^2}\right)^2 + \left(\frac{x_1}{z_1^2} + \frac{x_2}{z_2^2}\right)^2$$ is the equivalent reactance of the circuit. If the resistances and reactances of the two branched circuits are equal, the above values reduce to

$$r_0 = \tfrac{1}{2}r, \qquad x_0 = \tfrac{1}{2}x.$$

Example. —

$x_1 = 5$ ohms, $r_1 = 3$ ohms, $x_2 = 3$ ohms, $r_2 = 5$ ohms.
$z_1 = z_2 = 5.83$, $E = 500$ volts.

$$I_1 = \frac{E}{z_1} = 85.8 \text{ amp.},$$

$$I_2 = \frac{E}{z_2} = 85.8 \text{ amp.},$$

$$\phi_1 = \tan^{-1}\frac{x_1}{r_1} = 59^\circ\, 3',$$

$$\phi_2 = \tan^{-1}\frac{x_2}{r_2} = 31^\circ,$$

$$x_0 = \frac{\frac{5+3}{34}}{(\frac{8}{34})^2 + (\frac{8}{34})^2} = \frac{17}{8}, \quad r_0 = \frac{\frac{8}{34}}{(\frac{8}{34})^2 + (\frac{8}{34})^2} = \frac{17}{8},$$

$$I_0 = \frac{500}{\sqrt{(\frac{17}{8})^2 + (\frac{17}{8})^2}} = 166.4 \text{ amp.},$$

$$\phi_0 = \tan^{-1}\frac{x_0}{r_0} = \tan^{-1} 1 = 45°.$$

The current in the main circuit is less than the sum of the currents in the two branches, and this is because the phase angles in the two branches are different.

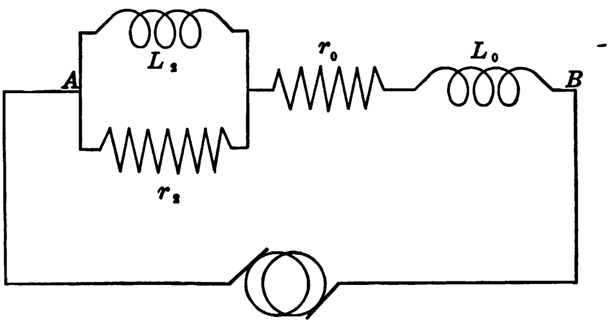

FIG. 37.

Two Branch Circuits, One having Resistance Only and the Other Inductance Only.—One branched circuit has a resistance r_2 and the other an inductance L_1 of negligible resistance, the main circuit inductance and resistance being r_0 and L_0 (see Fig. 37). The total impedance of circuit $A - B$ is given by the following expression:

$$Z = \sqrt{\left\{\frac{r_2\omega^2 L_2^2}{r_2^2 + \omega^2 L_2^2} + r_0\right\}^2 + \left\{\frac{L_2 r_2^2}{r_2^2 + \omega^2 L_2^2} + L_0\right\}^2}. \qquad (17)$$

This is a maximum when

$$r_2 = \frac{\omega^2 L_1 (L_2 + 2 L_0)}{2 r_0} + \frac{\omega L_2}{2 r_0}\{\omega^2 (L_2 + 2 L_0^2) + 4 r_0^2\}^{\frac{1}{2}}. \qquad (18)$$

For the value of r_2 given by equation (18), the current in the main circuit is a minimum. Hence shunting an inductance coil with a resistance sometimes increases the apparent resistance of the circuit, a result which has been noticed in practical work.

Example. —

$L_0 = L_2 = 0.1$ henry, $r_0 = 50$ ohms, $r_2 = 129$ ohms, $\omega = 2\pi \times 60$.

By (17) we have

$$\begin{aligned} Z^2 &= (68.1)^2 + (61.75)^2 = 8450.67, \\ Z &= 91.9 \text{ ohms.} \end{aligned}$$

If the inductance L_2 were not shunted by any resistance, the impedance of the circuit would be

$$\begin{aligned} Z^2 &= \omega^2 (L_1 + L_0)^2 + r_0^2 = (75.4)^2 + (50)^2 = 8185.16, \\ Z &= 90.5 \text{ ohms.} \end{aligned}$$

Shunting the inductance by a resistance increases the impedance of the circuit instead of decreasing as we would ordinarily expect.

A System of Branched Circuits in Parallel Each having Resistance and Inductance. — Let z_1, z_2, z_3 . . . denote the impedance of the respective branches and z_0 the total impedance of the circuit. The currents in the respective branches and the main circuit are as follows:

$$\left.\begin{aligned} I_1 &= \frac{E}{z_1} \cos (\omega t - \phi_1), \\ I_2 &= \frac{E}{z_2} \cos (\omega t - \phi_2), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ I_n &= \frac{E}{z_n} \cos (\omega t - \phi_n), \\ I_0 &= \frac{E}{z_0} \cos (\omega t - \phi_0), \\ z_0 &= \sqrt{r_0^2 + x_0^2}, \end{aligned}\right\} \qquad (19)$$

where

$$r_0 = \frac{\frac{1}{a}}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \text{equivalent resistance,}$$

$$x_0 = \frac{\frac{1}{b}}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \text{equivalent reactance,}$$

and

$$\frac{1}{a} = \sum_{k=1}^{k=n} \frac{r_k}{r_k^2 + x_k^2}, \quad \tan\phi_1 = \frac{x_1}{r_1}, \quad \tan\phi_2 = \frac{x_2}{r_2} \cdots$$

$$\frac{1}{b} = \sum_{k=1}^{k=n} \frac{x_k}{r_k^2 + x_k^2}, \quad \tan\phi_0 = \frac{x_0}{r_0}.$$

n = the number of branched circuits in parallel.

If the resistances and inductances respectively of all the branched circuits are equal,

$$\frac{1}{a} = n\left(\frac{r}{r^2 + x^2}\right),$$

$$\frac{1}{b} = n\left(\frac{x}{r^2 + x^2}\right),$$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{n^2(r^2 + x^2)}{(r^2 + x^2)^2} = \frac{n^2}{r^2 + x^2},$$

and

$$r_0 = \frac{r}{n},$$

$$x_0 = \frac{x}{n},$$

and

$$I_0 = \frac{E}{\sqrt{\frac{1}{n^2}(r^2 + x^2)}} \cos(\omega t - \phi_0), \tag{20}$$

that is, the main current is n times the current in any of the branches.

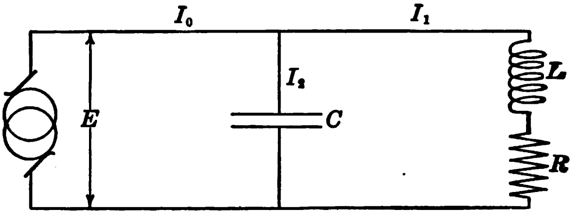

FIG. 38.

Inductance and Resistance Shunted by Condenser.* — Let I_1 denote the current in inductance branch, I_2 the current in capacity branch and I_0 the main current and $E \cos \omega t$ the impressed electromotive force. The currents in the two branches and the main circuit are

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* See "Theory and Calculations of Alternating Current Phenomena," 3d edition, Ch. VIII, by C. P. Steinmetz.

$$\left.\begin{aligned} I_1 &= \frac{E \cos(\omega t - \phi)}{\sqrt{x_m^2 + r^2}}, \\ I_2 &= E x_c \cos \omega t, \\ I_0 &= E \sqrt{\left(\frac{1}{x_c} - \frac{x_m}{x_m^2 + r^2}\right)^2 + \left(\frac{r}{r^2 + x_m^2}\right)^2} \cos(\omega t - \psi), \end{aligned}\right\} \quad (21)$$

$$\tan \psi = \frac{\dfrac{1}{x_c} - \dfrac{x_m}{x_m^2 + r^2}}{\dfrac{r}{r^2 + x_m^2}} = \frac{\dfrac{1}{x_c}(x_m^2 + r^2 - x_m x_c)}{r}. \quad (22)$$

$\frac{1}{2\pi fC} = x_c$

$2\pi fL = x_m$

When

$$\frac{1}{x_c} = \frac{x_m}{x_m^2 + r^2} \quad (23)$$

the circuit is in resonance, that is, the main current is in phase with the electromotive force and

$$I_0 = \frac{Er}{r^2 + x_m^2} \cos \omega t. \quad (24)$$

From (23) it is obvious that for this arrangement the resonance condition depends not only on the inductance and capacity, but also on the resistance. For every increase in r, x_c must be increased and the capacity decreased to maintain the resonance condition.

By varying x_c we can maintain the current in the main circuit in phase with the e.m.f. for any variation in the resistance of the receiving circuit. As an illustration we assume a value of $x_m = 15$, and calculate the values of x_c required to maintain the resonance condition in the circuit for values of r from 1 to 10 ohms. The results are plotted in the curve given in Fig. 39.

The e.m.f. E_0 across the resistance r is

$$E_0 = \frac{Er}{\sqrt{x_m^2 + r^2}} \cos(\omega t - \phi). \quad (25)$$

It is also to be noted that in such an arrangement of circuits, at resonance condition, the currents in the branches are larger than the currents in main circuit.

Example. —

$$x_m = 25, \quad r = 10, \quad \frac{1}{x_c} = \frac{25}{(25)^2 + (10)^2} = 0.0345, \quad E = 100 \text{ volts},$$

$$I_1 = \frac{100}{\sqrt{625 + 100}} = 3.7 \text{ amp.},$$

$$I_2 = 100 \times 0.0345 = 3.45 \text{ amp.},$$

$$I_0 = \frac{100 \times 10}{625 + 100} = 1.4 \text{ amp.}$$

The currents in the branch circuits are considerably larger than the current in the main circuit.

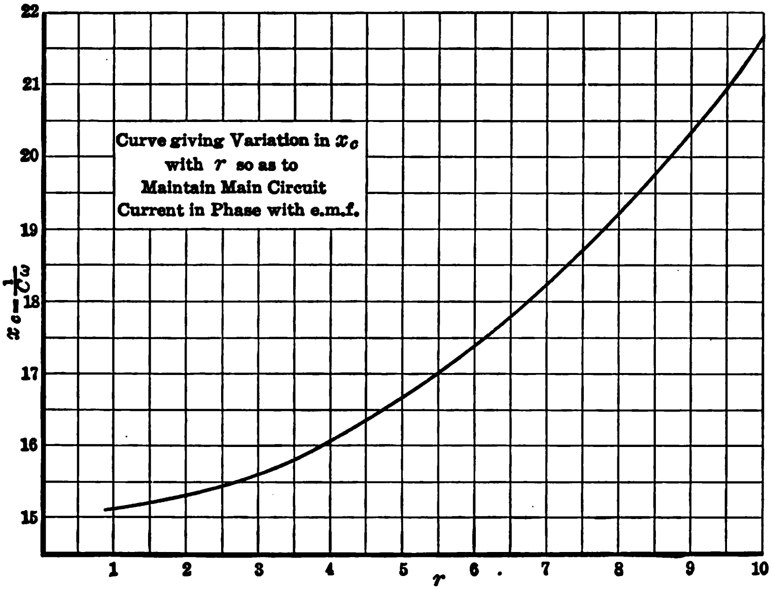


FIG. 39.

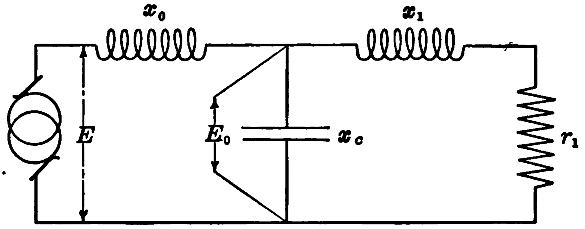


FIG. 40.

Inductive Load Shunted by Condenser. — Assume an electromotive force of sine form impressed on the circuit and a reactance x_0 in series with alternator (see Fig. 40). Denote by I_1 the current in resistance branch, I_2 the current in condenser, I_0 the main current, E the voltage of alternator and E_0 the voltage on the receiving circuit.

The distribution of the currents and voltage are as follows:

$$\left.\begin{aligned}
I_1 &= \frac{Ex_c}{\sqrt{\{x_c(x_0+x_1)-x_1x_0\}^2+r_1^2(x_c-x_0)^2}}\cos(\omega t-\phi_1),\\
\tan\phi_1 &= \frac{x_c(x_0+x_1)-x_1x_0}{r_1(x_c-x_0)},\\
I_2 &= -\frac{E\sqrt{r_1^2+x_1^2}}{\sqrt{\{x_c(x_0+x_1)-x_1x_0\}^2+r_1^2(x_c-x_0)^2}}\cos(\omega t-\phi_1+\psi).\\
\tan\psi &= \frac{x_1}{r},\\
I_0 &= \frac{E\sqrt{(x_c-x_1)^2+r_1^2}}{\sqrt{\{x_c(x_0+x_1)-x_1x_0\}^2+r_1^2(x_c-x_0)^2}}\cos(\omega t-\phi_1+\gamma),\\
\tan\gamma &= \frac{r}{x_1-x_c},\\
E_0 &= \frac{Ex_c\sqrt{r_1^2+x_1^2}}{\sqrt{\{x_c(x_0+x_1)-x_1x_0\}^2+r_1^2(x_c-x_0)^2}}\cos(\omega t-\phi_1+\psi).
\end{aligned}\right\}\quad(26)$$

If we make $x_c = x_1 = x_0 = x$ the denominators in the above equation reduce to x and the equations simplify to

$$\left.\begin{aligned}
I_1 &= \frac{E}{x}\sin\omega t,\\
I_2 &= \frac{E\sqrt{r_1^2+x^2}}{x^2}\sin(\omega t+\psi),\\
I_0 &= \frac{Er_1}{x^2}\cos\omega t,\\
E_0 &= \frac{E\sqrt{r_1^2+x^2}}{x}\sin(\omega t+\psi).
\end{aligned}\right\}\quad(27)$$

From these equations we can draw the interesting conclusion that for a constant value of E the current in the receiving circuit I_1 is constant, and independent of any variation in the load r_1. In other words this arrangement of circuits acts like a transformer converting a constant potential into a constant current.

Circuits in Parallel. — Two branch circuits in parallel, one having inductance and resistance and the other inductance capacity and resistance (see Fig. 41). Denote the current in branch (1) by I_1, that in branch (2) by I_2 and the current in main circuit by I_0; then

$$\left.\begin{array}{l}
I_1 = \dfrac{E}{\sqrt{x_1^2 + r_1^2}} \cos(\omega t - \phi), \\
\tan\phi = \dfrac{x_1}{r_1}, \\
I_2 = \dfrac{E}{\sqrt{x_2^2 + r_2^2}} \cos(\omega t - \psi), \\
\tan\psi = \dfrac{x_2}{r_2}, \\
I_0 = \\
E\sqrt{\left(\dfrac{r_1}{r_1^2+x_1^2}+\dfrac{r_2}{r_2^2+x_2^2}\right)^2+\left(\dfrac{x_1}{r_1^2+x_1^2}+\dfrac{x_2}{r_2^2+x_2^2}\right)^2}\cos(\omega t-\gamma), \\
\tan\gamma = \dfrac{\dfrac{x_1}{r_1^2+x_1^2}+\dfrac{x_2}{r_2^2+x_2^2}}{\dfrac{r_1}{r_1^2+x_1^2}+\dfrac{r_2}{r_2^2+x_2^2}}.
\end{array}\right\} \quad (28)$$

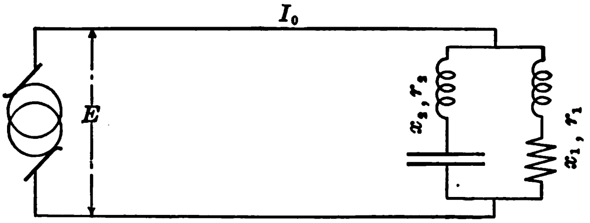


FIG. 41.

The circuit is in resonance when

$$\frac{x_1}{r_1^2 + x_1^2} + \frac{x_2}{r_2^2 + x_2^2} = 0. \qquad (29)$$

Putting for brevity $\frac{x_1}{r_1^2 + x_1^2} = k$, equation (29) becomes

$$kx_2^2 + kr_2^2 + x_2 = 0,$$

and

$$x_2 = \frac{-1 \pm \sqrt{1 - 4k^2r_2^2}}{2k}. \qquad (30)$$

The resonance condition depends on the resistance of the branch circuits as well as their inductances and capacities. If the constants of the circuits are of such magnitudes as to make $4k^2r_2^2$ greater than unity, the value of x_2 as given by (30) is an imaginary quantity, that is to say it is not possible to produce resonance in the circuit. If $4k^2r_2^2$ is less than unity, x_2 has two distinct values. The system is double periodic; for the same constants we may have resonance at two different frequencies.

As an illustration, let $k = 0.1$, $r_2 = 5$.

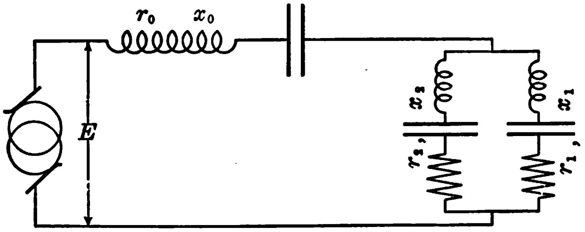
$$x_2 = \frac{-1 \pm \sqrt{1 - 0.2}}{2 \times 0.1} = -5 \text{ or } -95;$$

that is, if the frequencies are such as to make $x_2 = -5$ or -95, the circuit is in resonance and the current in the main circuit is

$$I_0 = E\left(\frac{r_1}{r_1^2 + x_1^2} + \frac{r_2^2}{r_2^2 + x_2^2}\right)\cos \omega t. \tag{31}$$

The Branch Circuits as Well as the Main Circuit having Inductance, Capacity and Resistance.—The currents in the branch circuits are denoted by I_1 and I_2 respectively, and the current in the main circuit by I_0.

$$\left.\begin{aligned} I_1 &= \frac{E\sqrt{r_2^2 + x_2^2}}{\sqrt{c^2 + d^2}}\cos(\omega t - \phi + \psi_1), \\ I_2 &= \frac{E\sqrt{r_1^2 + x_1^2}}{\sqrt{c^2 + d^2}}\cos(\omega t - \phi + \psi_2), \\ I_0 &= \frac{E\sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}}{\sqrt{c^2 + d^2}}\cos(\omega t - \phi + \psi_0), \end{aligned}\right\} \tag{32}$$



$$x_0 = L_0\,\omega - \frac{1}{C_0\omega}$$
$$x_1 = L_1\,\omega - \frac{1}{C_1\omega}$$
$$x_2 = L_2\,\omega - \frac{1}{C_2\omega}$$

FIG. 42.

where

$$\tan\phi = \frac{d}{c}, \quad \tan\psi_1 = \frac{x_2}{r_2}, \quad \tan\psi_2 = \frac{x_1}{r_1}, \quad \tan\psi_0 = \frac{x_1 + x_2}{r_1 + r_2}$$

and

$$c = r_0(r_1 + r_2) + r_1r_2 - x_0(x_1 + x_2) - x_1x_2,$$
$$d = r_1(x_0 + x_2) + r_2(x_1 + x_0) + r_0(x_1 + x_2).$$

When the inductances and capacities are so adjusted that each branch is separately in resonance, we get

$$x_1 = x_2 = x_0 = 0,$$
$$c = r_0 (r_1 + r_2) + r_1 r_2,\ d = 0,$$
$$\phi = \psi_1 = \psi_2 = \psi_0 = 0,$$

and

$$\left.\begin{aligned} I_1 &= \frac{E r_2}{r_0 (r. + r_2) + r_1 r_2} \cos \omega t, \\ I_2 &= \frac{E r_1}{r_0 (r_1 + r_2) + r_1 r_2} \cos \omega t, \\ I_0 &= \frac{E (r_1 + r_2)}{r_0 (r_1 + r_2) + r_1 r_2} \cos \omega t. \end{aligned}\right\} \qquad (33)$$

Mutual Inductance Between the Branched Circuits.* — We shall now consider the case of two branched circuits, in which

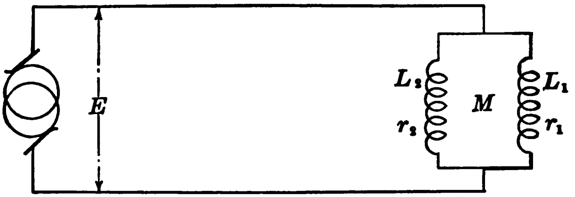


FIG. 43.

there is also mutual inductance between the two branches. Denoting the currents in the branched circuits by I_1 and I_2 respectively and the current in the main circuit by I_0 we have the following formulæ for the currents in the circuits:

$$\left.\begin{aligned} I_1 &= \frac{E\sqrt{(L_2 - M)^2\omega^2 + r_2^2}}{\sqrt{\{\omega^2(M^2 - L_1L_2) + r_1r_2\}^2 + \omega^2(L_1r_2 + L_2r_1)^2}} \cos(\omega t - \phi + \psi), \\ I_2 &= \frac{E\sqrt{(L_1 - M)^2\omega^2 + r_1^2}}{\sqrt{\{\omega^2(M^2 - L_1L_2) + r_1r_2\}^2 + \omega^2(L_1r_2 + L_2r_1)^2}} \cos(\omega t - \phi + \psi_2), \\ I_0 &= \frac{E\sqrt{(L_1 + L_2 - 2M)^2\omega^2 + (r_1 + r_2)^2}}{\sqrt{\{\omega^2(M^2 - L_1L_2) + r_1r_2\}^2 + \omega^2(L_1r_2 + L_2r_1)^2}} \cos(\omega t - \phi + \psi_0), \end{aligned}\right\} \quad (34)$$

$$\tan \phi = \frac{\omega (L_1 r_2 + L_2 r_1)}{\omega^2 (M^2 - L_1 L_2) + r_1 r_2}, \quad \tan \psi_1 = \frac{\omega (L_1 - M)}{r_2},$$

$$\tan \psi_2 = \frac{\omega (L_1 - M)}{r_1}, \quad \tan \psi_0 = \frac{\omega (L_1 + L_2 - 2M)}{r_1 + r_2.}.$$

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* J. J. Thomson, "Recent Researches in Electricity and Magnetism," Ch. VI; Lord Rayleigh, *Phil. Mag.*, May, 1886.

It is interesting to note that in such an arrangement of circuits, the currents in the branched circuits may, under some circumstances, be considerably larger than the current in the main circuit. Considering only the amplitudes of the currents, we have

$$\frac{I_1}{I_0} = \frac{\sqrt{(L_2 - M)^2\omega^2 + r_2^2}}{\sqrt{(L_1 + L_2 - 2M)^2\omega^2 + (r_1 + r_2)^2}}, \tag{35}$$

$$\frac{I_2}{I_0} = \frac{\sqrt{(L_1 - M)^2\omega^2 + r_1^2}}{\sqrt{(L_1 + L_2 - 2M)^2\omega^2 + (r_1 + r_2)^2}}. \tag{36}$$

If we neglect the resistances of the two branches compared with the reactances we have

$$\frac{I_1}{I_0} = \frac{L_2 - M}{L_1 + L_2 - 2M} \quad \text{and} \quad \frac{I_2}{I_0} = \frac{L_1 - M}{L_1 + L_2 - 2M}. \tag{37}$$

As an extreme case we may take the coefficient of coupling equal to unity, that is, $M = \sqrt{L_1L_2}$, and

$$\left.\begin{aligned} \frac{I_1}{I_0} &= \frac{L_2 - \sqrt{L_1L_2}}{L_1 + L_2 - 2\sqrt{L_1L_2}} = \frac{\sqrt{L_2}}{\sqrt{L_2} - \sqrt{L_1}}, \\ \frac{I_2}{I_0} &= \frac{L_1 - \sqrt{L_1L_2}}{L_1 + L_2 - 2\sqrt{L_1L_2}} = \frac{\sqrt{L_1}}{\sqrt{L_1} - \sqrt{L_2}}. \end{aligned}\right\} \tag{38}$$

If L_1 and L_2 are not much different from each other the denominators in equations (38) are very small and consequently the ratios $\frac{I_1}{I_0}$ and $\frac{I_2}{I_0}$ are large; that is, the currents in the branch circuits are considerably larger than the current in the main circuit.

As an illustration let us take the following constants:

$$L_1 = 0.1 \text{ henry}, \quad L_2 = 0.01 \text{ henry}, \quad M = 0.03 \text{ henry}.$$

$$\frac{I_1}{I_0} = \frac{0.02}{0.05} = 0.4, \quad \frac{I_2}{I_0} = \frac{0.07}{0.05} = 1.4.$$

The current in branch 2 is 1.4 times the current in the main circuit.

Power Factor.* — If an alternating e.m.f.,

$$e = E\sqrt{2}\cos\omega t,$$

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* See C. P. Steinmetz "Theory and Calculation of Alternating Current Phenomena," 3d edition, Ch. XXVII; also "Theorie der Wechselströme," von J. L. LaCour und O. S. Bragstad, Ch. XII.

produces in a circuit a current,

$$i = I\sqrt{2}\cos(\omega t \pm \phi),$$

where ϕ is the phase angle lagging or leading, the power

$$p = ei = EI\{\cos\phi + \cos(2\,\omega t \pm \phi)\}. \tag{39}$$

The average value of the power is

$$\left.\begin{aligned} P = EI\cos\phi &= I^2 r, \\ \cos\phi &= \frac{Ir}{E}. \end{aligned}\right\} \qquad 40)$$

The factor $\cos\phi$ is called the power factor.

Substituting (40) in (39) we get

$$p = P\left\{1 + \frac{\cos(2\,\omega t \pm \phi)}{\cos\phi}\right\}. \tag{41}$$

In an ordinary alternating current circuit the power fluctuates between the values

$$P\left\{1 + \frac{1}{\cos\phi}\right\} \quad \text{and} \quad P\left\{1 - \frac{1}{\cos\phi}\right\}. \tag{42}$$

When the phase angle is zero, $\cos\phi = 1$, the fluctuation of the power is between $2\,P$ and 0. If the phase angle is 90 degrees,

$$p = P\,(1 + \cos 2\,\omega t),$$

and the average or effective power $P = 0$.

If the pressure curve is not a simple harmonic curve but contains also harmonics, the average power supplied to the circuit is given by expression,

$$\begin{aligned} P &= E_1 I_1 \cos\phi_1 + E_3 I_3 \cos\phi_3 + E_5 I_5 \cos\phi_5 + \cdots \\ &= EI\cos\phi, \end{aligned} \tag{43}$$

where E and I are the effective values of the e.m.f. and the current, and

$$\cos\phi = \frac{r\sqrt{\dfrac{E_1^2}{r^2 + (x_m - x_c)^2} + \dfrac{E_3^2}{r^2 + \left(3\,x_m - \dfrac{x_c}{3}\right)^2} + \cdots}}{\sqrt{E_1^2 + E_3^2 + \cdots}}, \tag{44}$$

or we may put

$$\cos\phi = \sqrt{\frac{\cos^2\phi_1 + \left(\dfrac{E_3}{E_1}\right)^2\cos^2\phi_3 + \left(\dfrac{E_5}{E_1}\right)^2\cos^2\phi_5 + \cdots}{1 + \left(\dfrac{E_3}{E_1}\right)^2 + \left(\dfrac{E_5}{E_1}\right)^2 + \cdots}}, \tag{45}$$

where ϕ_1, ϕ_3, ϕ_5, etc., are the phase angles corresponding to the different harmonics.

Power Transmission.* — In alternating current transmission problems account must be taken of the line reactance which is of great importance; the efficiency of transmission and the voltage regulation depend on the electrical constants of the line. In the complete theory of long-distance transmission we must allow for the effect of the distributed capacity and inductance of the line. This is worked out fully in Chapter VI. In this section we shall assume that the resistance, inductance and capacity of the line are localized; that is, we replace the total inductance and capacity of the line by an inductance coil and condenser of equivalent value. This method gives fairly accurate results for short transmission lines, but for long lines the more complete formulæ given in Chapter VI will have to be used.

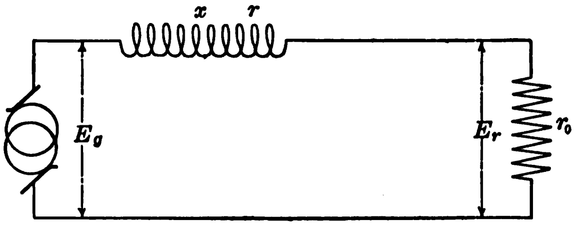

FIG. 44.

As a first approximation we shall neglect the capacity of the line entirely, and we shall also assume that the load is non-inductive.

Let

x = reactance of line,
r = resistance of line,
r_0 = resistance of receiving circuit,
E_g = e.m.f. at generator,
E_r = e.m.f. at receiving circuit,
I = current.

The current in the circuit is given by the following formula:

$$I = \frac{E_g}{\sqrt{x^2 + (r + r_0)^2}} = \frac{E_r}{r_0}. \tag{46}$$

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* See C. P. Steinmetz, "Theory and Calculation of Alternating Current Phenomena," Chapter IX; also "Theorie der Wechselströme," von J. L. La Cour and O. S. Bragstad, Ch. IV.

The ratio of e.m.f.'s at receiving circuit and generator is

$$\alpha = \frac{E_r}{E_g} = \frac{r_0}{\sqrt{x^2 + (r + r_0)^2}}. \tag{47}$$

The power delivered in the non-inductive receiving circuit is

$$P = E_r I = \frac{E_g{}^2 r_0}{x^2 + (r + r_0)^2}. \tag{48}$$

If E, x and r are fixed, the power supplied varies with r_0 and it has a maximum value when

$$r_0 = \sqrt{x^2 + r^2} = z. \tag{49}$$

Substituting the value of r_0 from (49) into (48) we get

$$P_{\max} = \frac{E_g{}^2}{2\,(z + r)}. \tag{50}$$

For maximum power supply at the receiving circuit the ratio of e.m.f. at receiver and generator is given by

$$\alpha_{\max} = \frac{E_r}{E_g} = \frac{1}{\sqrt{2\left(1 + \frac{r}{z}\right)}}. \tag{51}$$

The total power supplied by generator

$$P_t = E_g I \cos\phi = \frac{E_g{}^2}{2\,z}. \tag{52}$$

The efficiency for maximum power supplied to receiving circuit is

$$\eta = \frac{P_{\max}}{P_t} = \frac{z}{z + r}. \tag{53}$$

For comparison we will give here a set formulæ for a non-inductive line corresponding to equations (46) to (53), which are as follows:

$$I = \frac{E_g}{r + r_0} = \frac{E_r}{r_0}, \tag{54}$$

$$\alpha = \frac{E_r}{E_g} = \frac{r_0}{r + r_0}. \tag{55}$$

Power supplied to receiving circuit is

$$P = E_r I = \frac{E_g{}^2 r}{(r + r_0)^2} \tag{56}$$

and it is a maximum when $r = r_0$.

$$\left.\begin{aligned} P_{\max} &= \frac{E_g^2}{4r}, \\ \alpha_{\max} &= \tfrac{1}{2}. \end{aligned}\right\} \qquad (57)$$

Total power supplied is

$$P_t = \frac{E_g^2}{r + r_0} = \frac{E_g}{2r}. \qquad (58)$$

The efficiency for maximum power supply to receiving circuit is

$$\eta = \frac{P_{\max}}{P_t} = \frac{1}{2}.$$

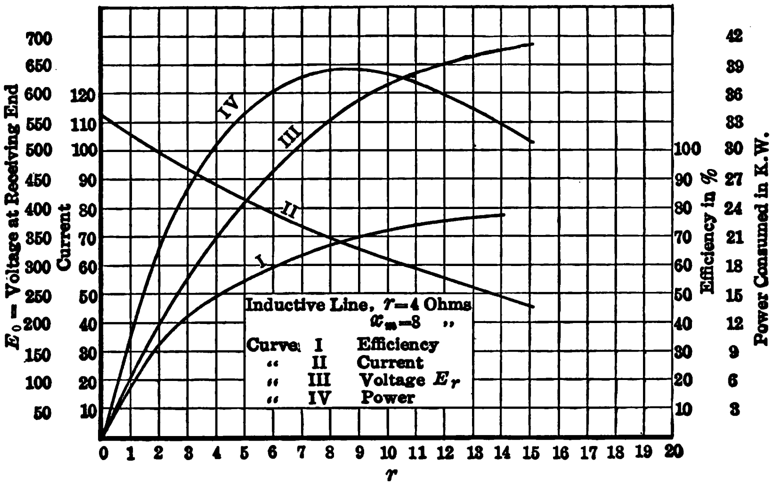

FIG. 45. — Power transmission characteristic curves.

As an illustration we give here a set of curves showing the variation with the load of I, E_r, η and P_r for an inductive line of the following constants:

$$E = 1000 \text{ volts}, \quad r = 4 \text{ ohms}, \quad x = 8.$$

Inductive Load. — Let x_1, r_1 represent the reactance and resistance of transmission line, x_2, r_2 the reactance and resistance of receiving circuit, E_g = e.m.f. of generator, E_r = e.m.f. at receiving circuit.

$$I = \frac{E_g}{\sqrt{(x_1 + x_2)^2 + (r_1 + r_2)^2}} = \frac{E_r}{\sqrt{x_2^2 + r_2^2}}. \qquad (59)$$

The phase angle ϕ_1 between E_g and I is given by

$$\tan \phi_1 = \frac{x_1 + x_2}{r_1 + r_2}$$

and the phase angle ϕ_2 between E_r and I is given by

$$\tan \phi_2 = \frac{x_2}{r_2}.$$

We may put equation (59) in the following form:

$$I = E_g \sqrt{\frac{g_2^2 + b_2^2}{(1 + r_1g_2 + x_1b_2)^2 + (x_1g_2 - r_1b_2)^2}} \tag{60}$$

and

$$\alpha = \frac{E_r}{E_g} = \frac{1}{\sqrt{(1 + r_1g_2 + x_1b_2)^2 + (x_1g_2 - r_1b_2)^2}}, \tag{61}$$

where

$$g_2 = \frac{r_2}{r_2^2 + x_2^2} = \text{effective conductance of receiving circuit},$$

$$b_2 = \frac{x_2}{r_2^2 + x_2^2} = \text{effective susceptance of receiving circuit}.$$

The power supplied to receiving circuit,

$$P = E_r^2 g_2 = E_g^2 g_2 \alpha^2, \tag{62}$$

and it has its maximum value when

$$g_2 = \sqrt{g_1^2 + (b_1 + b_2)^2}; \tag{63}$$

and for this value of g_2 we have

$$\alpha_{\max} = \frac{1}{\sqrt{2\, g_2\, (g_2 z_1^2 + r_1)}}. \tag{64}$$

The maximum power that can be supplied to the receiving circuit is

$$P_{\max} = \frac{E_g^2}{2\, (g_2 z_1^2 + r_1)}. \tag{65}$$

For constant susceptance b_2 the efficiency has its maximum value when $g_2 = b_2$ and

$$\eta_{\max} = \frac{1}{1 + 2\, r_1\, b_2}. \tag{66}$$

If we do not neglect the capacity of the line, the problem becomes somewhat more complex, but it may be simplified very materially by the following considerations:

In engineering practice, power transmission problems frequently present themselves in the following way. Given the values of the voltage and current and their phase relation at the receiving circuit, what will be the corresponding values of voltage and current at the generator end, the power transmitted, efficiency, etc.? And *vice versa,* if the e.m.f. and current are given at the generator end what will be the corresponding values at the receiving end? In other words given the constants of the line and electrical conditions at one end of the line to determine all the other factors of the problem.

As stated before, the exact solution of the problem necessitates taking into account the effect of the distributed inductance and capacity of the line which are worked out in Chapter VI. For short lines, however, we can get fairly accurate results by assuming localized inductance and capacity.

Different formulæ may be obtained depending on the assumptions made regarding the capacity distribution, and we may have the following cases:

(1) Capacity of line neglected entirely.

(2) Entire capacity of line shunted across the middle of the line.

(3) Half of line capacity shunted across at each end of the line.

(4) Four-sixths of line capacity shunted across the middle of the line and one-sixth of line capacity at each end of the line.

We shall give here formulæ corresponding to the first three cases, and work out some examples to compare the closeness of approximation of the above three cases.* The fourth case was not considered for the reason that the formulæ are somewhat complex and the results obtained by this method are not more accurate than those obtained by the method considered in case (3). It is not considered advisable to multiply the number of formulæ unnecessarily.

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* Ch. P. Steinmetz, "Theory and Calculation of Alternating Current Phenomena," 3d edition, Chap. XIII.

F. A. C. Perrine and F. G. Baum, The Use of Aluminium Line Wire and Some Constants for Transmission Lines, *Trans. Am. Ins. E. E.,* May, 1900.

P. H. Thomas, Output and Regulation in Long Distance Lines, *Trans. Am. Ins. E. E.,* June, 1909.

P. H. Thomas, Calculation of High Tension Lines, *Trans. Am. Inst. E. E.,* June, 1909.

We shall use the following notation:

$$\begin{aligned} E_g &= E_g' \pm jE_g'' = \text{e.m.f. at generator,} \\ I_g &= I_g' \pm jI_g'' = \text{current at generator end,} \\ E_r &= E_r' \pm jE_r'' = \text{e.m.f. at receiving end of line,} \\ I_r &= I_r' \pm jI_r'' = \text{current at receiving end of line.} \end{aligned}$$

The double sign in the above notation is to indicate whether the phase angle is leading or lagging; the plus sign meaning phase angle leading, and the minus sign phase angle lagging. Suppose the voltage and current are given at the receiving end of the line and the phase angle of the current with respect to the voltage is leading, then,

$$\left.\begin{aligned} E_r &= E_r', \\ I_r &= I_r' + jI''. \end{aligned}\right\} \qquad (67)$$

If on substituting these values in the formulæ we obtain for the e.m.f. and current at generator end,

$$\begin{aligned} E_g &= E_g' - jE_g'', \\ I_g &= I_g + jI_g'', \end{aligned}$$

it would mean that the e.m.f. at generator end lags behind the e.m.f. at receiving end by an angle $\tan^{-1}\frac{E_g''}{E_g'}$, while the current at generator leads the e.m.f. at receiving end by an angle $\tan^{-1}\frac{I_g''}{I_g'}$.

Case I. Line capacity neglected entirely.

Suppose that the admittance of receiving circuit $g \pm jb$ and E_r are given, we have

$$I_r = E_r(g \pm jb) = I_r' \pm jI_r''.$$

E_g and I_g have the following values:

$$\left.\begin{aligned} I_g &= I_r, \\ E_g &= \{E_r \pm L\omega I_r'' + rI_r'\} + j\{L\omega I_r' \pm rI_r''\}. \end{aligned}\right\} \qquad (68)$$

Where a double sign appears in the above equation, the upper sign is to be used when the current at the receiver end is leading and the lower sign when the receiving current is lagging.

If the voltage and current are given at the generator end, that is E_g and $I_g' \pm jI_g''$ given, we have

$$\left.\begin{aligned} I_r &= I_g, \\ E_r &= \{E_g \pm L\omega I_g'' - rI_g'\} - j\{L\omega I_g' \pm rI_g''\}. \end{aligned}\right\} \qquad (69)$$

The double sign in the terms of equation (69) has the same significance as in (68).

Example.—Power to be transmitted over a three-phase line 50 miles long using hard-drawn stranded copper wire No. 000, triangularly spaced 10 feet spacing between wires. Frequency 60 cycles. At the receiving end we have the following conditions: Line voltage 30 kv. between wire and neutral, load current at receiving end 150 amperes at 90 per cent power factor, lagging current. Phase angle lagging 25° 50′.

Calculating the inductance and capacity of the line by formulæ given in Chapters II and III we find,

$$\begin{aligned} r &= 0.33 \text{ ohm per mile,} \\ L &= 2.13 \text{ mh. per mile,} \\ C &= 0.014 \text{ mf. per mile.} \end{aligned}$$

The total resistance, inductance and capacity of the line are 16.5 ohms, 0.1065 henry and 0.7 mf. respectively.

The electrical conditions at the receiving end are

$$\begin{aligned} E_r &= 30 \text{ kv.,} \\ I_r &= I_r' - jI_r'' = 135 - j\,65.4 \text{ amp.} \end{aligned}$$

In formula (68) we neglect the capacity of the line entirely, hence introducing the values of r, L and ω we get

$$\begin{aligned} I_g &= I_r = 135 - j\,65.4 \text{ amp.,} \\ E_g &= \{30 + 2.63 + 2.23\} + j\,\{5.42 - 1.08\} \\ &= 34.86 + j\,4.34. \end{aligned}$$

The absolute value of e.m.f. at generator is

$$E_g = \sqrt{E_g'^2 + E_g''^2} = 35.1 \text{ kv.}$$

The voltage at generator leads the voltage at receiver by an angle

$$\tan^{-1}\frac{4.34}{34.86} = 7^\circ\ 5';$$

hence the current at generator end lags behind the generator e.m.f. by the angle 25° 50′ + 7° 5′ = 32° 55′.

The power factor at the generator end is

$$\cos\,(32^\circ\ 55') = 0.84.$$

The efficiency of transmission

$$\eta = \frac{30 \times 150 \times 0.9}{35.1 \times 150 \times 0.84} = 91.6 \text{ per cent.}$$

Example. — The same conditions as in above example except that the line is 150 miles long.

We have as in the previous example,

$$I_g = I_r = 135 - j\,(65.4) \text{ amp.,}$$
$$E_g = 44.58 + j\,13.02 \text{ kv.}$$

The absolute value of e.m.f. at generator is

$$E_g = \sqrt{(44.58)^2 + (13.02)^2} = 46.4 \text{ kv.}$$

The generator e.m.f. leads the voltage at the receiving end by the angle

$$\tan^{-1}\frac{13.02}{44.58} = 16^\circ\ 16'.$$

The current at generator end lags behind E_g by the angle $25^\circ\ 50' + 16^\circ\ 16' = 42^\circ\ 6'$.

The power factor at generator end = cos (42° 6′) = 0.742.

The efficiency of transmission is

$$\eta = \frac{30 \times 150 \times 0.9}{46.4 \times 150 \times 0.742} = 78.4 \text{ per cent.}$$

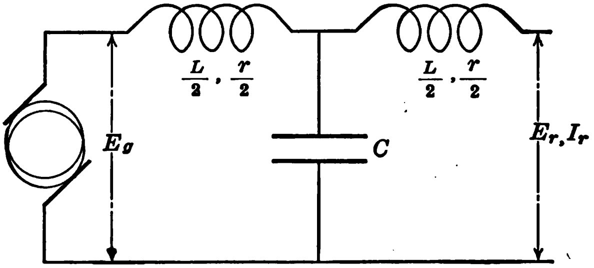

Fig. 46.

Case II. Capacity of the line shunted across the middle of the line.

If the admittance of receiving circuit $g \pm jb$ and the voltage E_r are given, we have $I_r = E_r\,(g \pm jb) = I_r' \pm jI_r''$.

The voltage and current at the generator end are given by the formulæ

$$I_g = \{KI_r' \mp \tfrac{1}{2}C\omega r I_r''\} + j\{\tfrac{1}{2}C\omega r I_r' \pm KI_r'' + E_r C\omega\} = I_g' \pm jI_g'', \quad (70)$$

$$E_g = E_r + \{\tfrac{1}{2}r\,(I_g' + I_r') - \tfrac{1}{2}L\omega\,(I_g'' \pm I_r'')\}$$
$$+ j\{\tfrac{1}{2}r\,(I_g'' \pm I_r'') + \tfrac{1}{2}L\omega\,(I_g' + I_r')\} = E_g' \pm jE_g'', \quad (71)$$

where $$K = 1 - \tfrac{1}{2}CL\omega^2.$$

The double signs in the terms of equations (70) and (71) have the same significance as in equations (68) and (69). The upper sign is to be used when I_r'' is positive, that is, the receiving current leading voltage at receiving end, and the lower sign is to be used when I_r'' is negative.

If the voltage and current are given at the generator end, that is, E_g and $I_g = I_g' \pm jI_g''$ are given, the current and voltage at the receiving end are given by the formulæ:

$$\begin{aligned} I_r &= \{KI_g' \mp \tfrac{1}{2} rC\omega I_g''\} + j\{\pm KI_g'' + \tfrac{1}{2} C\omega rI_g' - E_gC\omega\} \\ &= I_r' \pm jI_r'', \end{aligned} \tag{72}$$

$$\begin{aligned} E_r = E_g &- \{\tfrac{1}{2} r(I_g' + I_r') - \tfrac{1}{2} L\omega(I_r'' \pm I_g'')\} \\ &- j\{\tfrac{1}{2} r(I_r'' \pm I_g'') + \tfrac{1}{2} L\omega(I_g' + I_r')\} = E_g' \pm jE_g'', \end{aligned} \tag{73}$$

where as before $K = 1 - \tfrac{1}{2} CL\omega^2.$

As an illustration and for comparison we shall work out examples given above by formulæ (70) and (71).

$$r = 16.5 \text{ ohms}, \quad L = 0.1065 \text{ henry}, \quad C = 0.7 \text{ mf.}$$

$$K = 0.995, \quad C\omega = 264 \times 10^{-6}.$$

$$E_r = 30 \text{ kv.}, \quad I_r = 135 - j\,65.4 \text{ amp.}$$

The phase angle at the receiving end is 25° 51′ lagging.

$$I_g = \{134.3 + 0.14\} + j\{0.29 - 65.07 + 7.92\} = 134.44 - j\,56.86 \text{ amp.};$$

absolute value of $I_g = \sqrt{(134.44)^2 + (56.80)^2} = 146$ amp.

$$E_g = (30 + 2.22 + 2.45) + j(-1.01 + 5.41) = 34.67 + j\,4.4 \text{ kv};$$

absolute value,

$$E_g = \sqrt{(34.67)^2 + (4.4)^2} = 34.95 \text{ kv}.$$

The phase angle between I_g and E_r is $\tan^{-1} \dfrac{56.86}{134.4} = 32° 16'$, lagging.

The phase angle between E_g and E_r is $\tan^{-1} \dfrac{4.4}{34.67} = 7° 15'$, leading.

I_g lags behind E_g by the angle $23° 16' + 7° 15' = 30° 31'$.

Power factor at generating end = cos (30° 31′) = 0.862.

Efficiency of transmission,

$$\eta = \frac{150 \times 30 \times 0.9}{146 \times 34.95 \times 0.862} = 92.1 \text{ per cent.}$$

Example.—The same conditions as in the example above except that the length of the line is 150 miles long.

$$E_r = 30\,\text{kv.},\quad I_r = 135 - j\,65.4\ \text{amp.},\quad K = 0.952.$$

$$I_g = (128.52 + 1.26) + j\,(2.61 - 62.26 + 23.76) = 129.8 - j\,35.9\ \text{amp.}$$

Absolute value of current at receiving end,

$$I_g = \sqrt{(129.8)^2 + (35.9)^2} = 135\ \text{amp.}$$

$$E_g = (30 + 6.55 + 6.1) + j\,(-2.51 + 15.95) = 42.65 + j\,13.44\ \text{kv.}$$

Absolute value of voltage,

$$E_g = \sqrt{(42.65)^2 + (13.44)^2} = 44.7\ \text{kv.}$$

The phase angle between I_g and E_r is $\tan^{-1}\dfrac{35.9}{129.8} = 15^\circ\,30'$ lagging.

The phase angle between E_g and E_r is

$$\tan^{-1}\frac{13.44}{42.65} = 17^\circ\,30',\ \text{leading.}$$

I_g lags behind E_g by the angle $15^\circ\,30' + 17^\circ\,30' = 33^\circ$.
Power factor at generating end, $\cos(33^\circ) = 0.84$.

Efficiency of transmission

$$\eta = \frac{150 \times 30 \times 0.9}{135 \times 44.7 \times 0.84} = 0.80 = 80\ \text{per cent.}$$

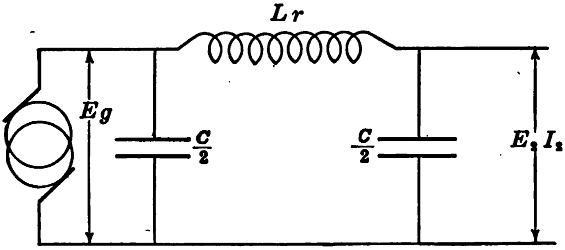

FIG. 47.

Case III. Half of line capacity shunted across at each end of the line.

As in previous cases *I* and *II* we shall assume that E_r and I_r are given in magnitude as well as phase relation, that is,

$$I_r = I_r' + j\,(\pm I_r''). \qquad (74)$$

The values of the voltage and current at the generator end are given by the formulæ

$$E_g = \{E_rK + rI_r' - L\omega I_r''\} + j\,\{\tfrac{1}{2}\,C\omega r\,E_r + rI_r'' + L\omega I_r'\}$$
$$= E_g' + j\,(\pm E_g'') \qquad (75)$$
$$I_g = \{I_r' - \tfrac{1}{2}E_g''C\omega\} + j\,\{\tfrac{1}{2}\,E_rC\omega + \tfrac{1}{2}E_g'C\omega + I_r''\} = I_g' \pm jI_g''. \qquad (76)$$

The values of I_r'' and E_g'' are to be taken positive or negative depending on the sign of these quantities in the parentheses in (74) and (75).

If E_g and I_g are given in magnitude as well as in phase relation,

$$I_g = I_g' + j\,(\pm I_g''). \qquad (77)$$

We have the following expressions for E_r and I_r:

$$E_r = \{E_gK - rI_g' + L\omega I_g''\} + j\,\{\tfrac{1}{2}\,E_g\,C\omega r - L\omega I_g' - rI_g''\}$$
$$= E_r' + j\,(\pm E_r''), \qquad (78)$$
$$I_r = \{I_g' + \tfrac{1}{2}E_r''C\omega\} + j\,\{I_g'' - \tfrac{1}{2}E_gC\omega - \tfrac{1}{2}E_r'C\omega\} = I_r' + j(\pm I_r''), \qquad (79)$$

where

$$K = 1 - \tfrac{1}{2}\,LC\omega^2.$$

In the following two examples we shall use the same data as in examples considered under Case II.

Example. — Fifty-mile transmission line.

$$r = 16.5 \text{ ohms.},$$
$$L = 0.1005,$$
$$C = 0.7 \text{ mf.},$$
$$K = 0.995,$$
$$E_r = 30 \text{ kv.}, \quad I_r = 135 - j\,65.4 \text{ amp.}$$
$$E_g = \{29.85 + 2.23 + 2.63\} + j\,\{0.065 - 1.08 + 5.42\}$$
$$= 34.7 + j\,4.4 \text{ kv.}$$

Absolute value of voltage is

$$E_g = \sqrt{(34.7)^2 + (4.4)^2} = 35 \text{ kv.}$$
$$I_g = \{135 - 0.58\} + j\,\{3.96 + 4.58 - 65.4\} = 134.4 - j\,56.86 \text{ amp.}$$

Absolute value of current is $I_g = \sqrt{(134.4)^2 + (56.86)^2} = 145.9$ amp.

$$E_g \text{ leads } E_r \text{ by the angle } \tan^{-1}\frac{4.4}{34.7} = 7^\circ\,14',$$
$$I_g \text{ leads } E_r \text{ by the angle } \tan^{-1}\frac{56.86}{134.42} = 22^\circ\,57',$$
$$I_g \text{ lags behind } E_g \text{ by the angle, } 22^\circ\,57' + 7^\circ\,14' = 30^\circ\,11'.$$

Power factor at generating end,

$$\cos(30^\circ\, 11') = 0.864.$$

Efficiency of transmission,

$$\eta = \frac{150 \times 30 \times 0.9}{145.9 \times 35 \times 0.864} = 91.8 \text{ per cent.}$$

Example. — One hundred fifty-mile three-phase transmission line.

$$\begin{aligned} R &= 49.5 \text{ ohms}, \\ L &= 0.3195, \\ C &= 0.21 \text{ mf.}, \\ K &= 0.952. \end{aligned}$$

$$E_r = 30 \text{ kv.}, \quad I_r = 135 - j\,65.4 \text{ amp.}$$

$$\begin{aligned} E_g &= \{28.5 + 6.7 + 7.9\} + j\,\{0.6 - 3.25 + 16.25\} \\ &= 43.1 + j\,13.6 \text{ kv.} \end{aligned}$$

The absolute value of the generator voltage,

$$E_g = \sqrt{(43.1)^2 + (13.6)^2} = 45.2 \text{ kv.}$$

$$I_g = \{135 - 5.4\} + j\,\{11.9 + 17.05 - 65.4\} = 129.6 - j\,36.5 \text{ amp.}$$

Absolute value of current,

$$I_g = \sqrt{(129.6)^2 + (36.5)^2} = 134.7 \text{ amp.}$$

E_g leads E_r by the angle $\tan^{-1}\dfrac{13.6}{34.1} = 17^\circ\, 30'$,

I_g lags behind E_r by the angle $\tan^{-1}\dfrac{36.5}{129.6} = 15^\circ\, 45'$,

I_g lags behind E_g by the angle $17^\circ\, 30' + 15^\circ\, 45' = 33^\circ\, 15'$.

The power factor at generator end,

$$\cos(33^\circ\, 15') = 0.836.$$

The efficiency of transmission,

$$\eta = \frac{150 \times 30 \times 0.9}{134.7 \times 45.2 \times 0.836} = 79.5 \text{ per cent.}$$

TABLE XVI

CONSTANTS OF LINE. — $r=16.5$ ohms, $L=0.1065$ henry, $C=0.7$ mf.

	Line capacity neglected entirely.		Line capacity shunted across middle of line.		Half line capacity shunted across each end of line.		Distributed capacity formula.	
	50 miles.	150 miles.	50 miles.	150 miles.	50 miles.	150 miles.	50 miles.	150 miles.
$E_r=$	30 kv.	30 kv.	30 kv.	30 kv.	30 kv.	30 kv.	30 kv.	30 kv.
$I_r=\sqrt{I_r'^2+I_r''^2}$ ...	150 amp.	150 amp.	150 amp.	150 amp.	150 amp.	150 amp.	150 amp.	150 amp.
$E_g=\sqrt{E_g'^2+E_g''^2}$.	35.1 kv.	46.4 kv.	34.95 kv.	44.7 kv.	35 kv.	45.2 kv.	35 kv.	44.8 kv.
$I_g=\sqrt{I_g'^2+I_g''^2}$ ..	150 amp.	150 amp.	146 amp.	135 amp.	145.9 amp.	134.7 amp.	145.9 amp.	135 amp.
Phase angle (E_r, I_r)	25° 50′ lag.	25° 50′ lag.	25° 50′ lag.	25° 50′ lag.	25° 50′ lag.	25° 50′ lag.	25° 50′ lag.	25° 50′ lag.
Phase angle (E_g, I_g)	32° 55′ lag.	42° 6′ lag.	30° 31′ lag.	33° lag.	30° 11′ lag.	33° 15′ lag.	30° 11′ lag.	32° 32′ lag.
Power factor at receiving end	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
Power factor at sending end	0.84	0.742	0.862	0.839	0.864	0.836	0.864	0.843
Efficiency.........	91.6%	78.4%	92.1%	80%	91.8%	79.5%	91.8%	79.4%

TRANSFORMERS

Any two electrical circuits which are magnetically interlinked so that when a current flows in one circuit it will induce an e.m.f. in the other circuit constitutes a transformer. For all commercial work the iron-core transformer is used. In high-frequency work, as in the case of wireless telegraphy for instance, air-core transformers are used exclusively.

We shall first give the equation governing the action of an air-core transformer, and then give a brief discussion of the theory of the iron-core transformer.

Air-core Transformer.* — Denote by L_1, R_1 and L_2, R_2 the total inductance and resistance of the primary and secondary circuit respectively and by M the mutual inductance between the two circuits. We shall also assume an e.m.f. of sine wave applied to the primary circuit.

The equations expressing the reactions in the transformer circuits are

$$\left.\begin{aligned} L_1\frac{dI_1}{dt} + R_1I_1 + M\frac{dI_2}{dt} &= Ee^{j\omega t}, \\ L_2\frac{dI_2}{dt} + R_2I_2 + M\frac{dI_1}{dt} &= 0, \end{aligned}\right\} \qquad (80)$$

from which we obtain the values of the currents in the two circuits as follows:

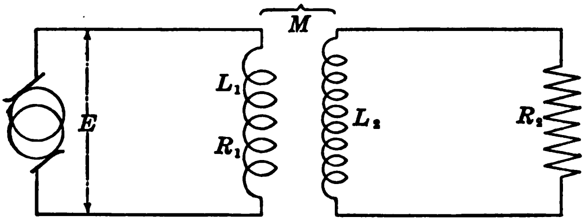

FIG. 48.

$$I_1 = \frac{E\cos(\omega t - \phi)}{\sqrt{\left\{R_1 + \frac{M^2\omega^2R_2}{z_2^2}\right\}^2 + \left\{L_1\omega - \frac{M^2\omega^2L_2\omega}{z_2^2}\right\}^2}}, \qquad (81)$$

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* Alex. Russell, "Theory of Alternating Currents," Vol. 1, Chap. X.
Clerk Maxwell, "A Dynamical Theory of the Electromagnetic Field," *Phil. Trans.*, 1865, p. 475.

$$\tan \phi = \frac{\omega \left(L_1 - \dfrac{M^2\omega^2 L_2}{z_2^2}\right)}{R_1 + \dfrac{M^2\omega^2 R_2}{z_2^2}},$$

$$I_2 = \frac{E\, M\omega \sin (\omega t - \phi - \gamma)}{z_2 \sqrt{\left\{R_1 + \dfrac{M^2\omega^2 R_2}{z_2^2}\right\}^2 + \left\{L_1\omega - \dfrac{M^2\omega^2 L_2\omega}{z_2^2}\right\}^2}}, \tag{82}$$

$$\tan \gamma = \frac{L_2\omega}{R_2}, \quad z_2^2 = L_2^2\omega^2 + R_2^2.$$

The value of the current in the primary circuit is the same as that in a simple circuit whose resistance is $R_1 + \frac{M^2\omega^2 R_2}{z_2^2}$ and inductance is $L_1 - \frac{M^2\omega^2 L_2}{z_2^2}$, that is, the secondary circuit influences the primary circuit in the same way as if its resistance were increased and the inductance decreased. In other words the effective resistance in $R_1 + \frac{M^2\omega^2 R_2}{z_2^2}$ instead of R_1 and effective inductance is $L_1 - \frac{M^2\omega^2 L_2}{z_2^2}$ instead of L_1. If we neglect the resistance of the secondary circuit, the effective inductance of the primary circuit is

$$L_1 - \frac{M^2\omega^2 L_2}{L_2^2\omega^2} = L_1\left(1 - \frac{M^2}{L_1 L_2}\right) = \sigma L_1. \tag{83}$$

The quantity σ is called the leakage factor.

Resonance Transformer.* — If we introduce capacities in the primary and secondary circuits, the expressions for the currents have exactly the same form as that given by equations (81) and (82) except that $L_1\omega$ and $L_2\omega$ are replaced by the terms $L_1\omega - \frac{1}{C_1\omega}$ and $L_2\omega - \frac{1}{C_2\omega}$ where C_1 and C_2 are the capacities in primary and secondary circuits respectively. Now suppose the primary and secondary circuits are separately tuned so as to be in resonance for the frequency of the impressed e.m.f. Then

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* G. W. Pierce, "Theory of Electrical Oscillations in Coupled Circuits," *Proc. Am. Acad.*, Vol. 46, pp. 293–322, Jan., 1911.

J. S. Stone, "The Maximum Current in the Secondary of a Transformer," *Phys. Rev.*, Vol. 32, pp. 398–405, April, 1911.

G. Benischke, "Der Resonanztransformotor," E. T. Z., 1907, p. 25.

J. Bethenod, "Über den Resonanztransformotor," *Jahrbuch der Drahtlosen Telegraphie*, Vol. I, pp. 534–570.

$$L_1\omega - \frac{1}{C_1\omega} = 0, \quad L_2\omega - \frac{1}{C_2\omega} = 0, \quad Z_2 = R_2, \tag{84}$$

and equations (81) and (82) reduce to

$$\left.\begin{aligned} I_1 &= \frac{ER_2 \cos \omega t}{R_1R_2 + M^2\omega^2}, \\ I_2 &= \frac{EM\omega \sin \omega t}{R_1R_2 + M^2\omega^2}. \end{aligned}\right\} \tag{85}$$

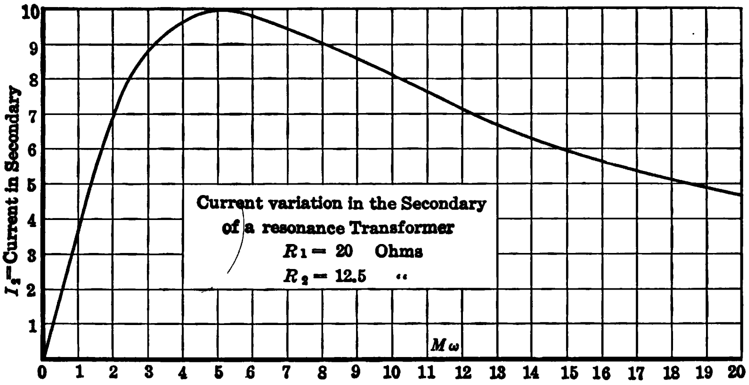

FIG. 49.—Current variation in the secondary of a resonance transformer for different degrees of coupling.

The current in the secondary circuit is a maximum when $M^2\omega^2 = R_1R_2$ and

$$I_{2,\max} = \frac{E \sin \omega t}{2\sqrt{R_1R_2}}. \tag{86}$$

If we have only a condenser in the secondary as shown in Fig. 50, we have the following expressions for the currents in the primary and secondary circuits and the voltage on the condenser.

$$I_1 = \frac{E\sqrt{R_2^2 + \left(L_2\omega - \frac{1}{C\omega}\right)^2} \cos(\omega t - \phi + \psi)}{\sqrt{\left\{(M^2 - L_1L_2)\omega^2 + \frac{L_1}{C} + R_1R_2\right\}^2 + \omega^2\left\{L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right\}^2}}, \tag{87}$$

$$\tan \phi = \frac{\omega\left(L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right)}{(M^2 - L_1L_2)\,\omega^2 + \frac{L_1}{C} + R_1R_2}, \quad \tan \psi = \frac{L_2\omega - \frac{1}{C\omega}}{R_2},$$

$$I_2 = \frac{E\,M\omega \sin(\omega t - \phi)}{\sqrt{\left\{(M^2 - L_1L_2)\omega^2 + \frac{L_1}{C} + R_1R_2\right\}^2 + \omega^2\left\{L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right\}^2}}. \tag{88}$$

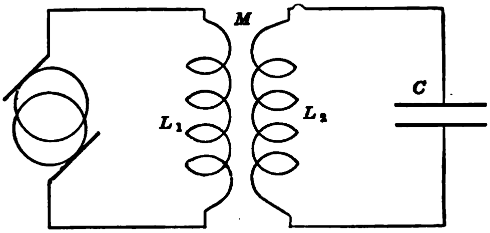

FIG. 50.

The voltage on condenser is given by

$$e = \frac{-E\frac{M}{C}\cos(\omega t - \phi)}{\sqrt{\left\{(M^2 - L_1L_2)\omega^2 + \frac{L_1}{C} + R_1R_2\right\}^2 + \omega^2\left\{L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right\}^2}}. \tag{89}$$

The condition for maximum currents in the circuits, that is, the resonance condition, is

$$(M^2 - L_1L_2)\,\omega^2 + \frac{L_1}{C} = 0,$$

or

$$1 - \frac{C}{L_1}(L_1L_2 - M^2)\,\omega^2 = 0.$$

$$\omega^2 = \frac{1}{CL_2(1 - K^2)}, \text{ where } K^2 = \frac{M^2}{L_1L_2}. \tag{90}$$

If the constants of the circuits are fixed so as to satisfy equation (90), and if we neglect R_1R_2 which is generally small, we have the following expressions for the currents and voltage in the circuits:

$$I_1 = \frac{E\sqrt{R_2^2 + \left(L_2\omega - \frac{1}{C\omega}\right)^2}}{\omega\left\{L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right\}}\sin(\omega t + \psi), \tag{91}$$

$$I_2 = \frac{-EM\omega\cos\omega t}{\omega\left\{L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right\}}\cos\omega t, \tag{92}$$

$$e = \frac{-EM\sin\omega t}{C\omega\left\{L_1R_2 + L_2R_1 - \frac{R_1}{C\omega^2}\right\}}, \qquad (93)$$

$$\tan\psi = \frac{L_2\omega - \frac{1}{C\omega}}{R_2}.$$

The form of transformer discussed above is used extensively in the production of high-potential high-frequency currents. The arrangement of circuits used is indicated in Fig. 51.

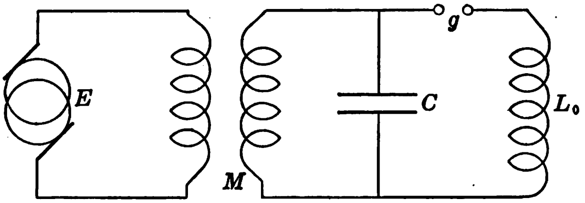

Fig. 51.

E is an alternator, M, a high potential transformer.

When the condenser C is charged to its maximum, a spark occurs at the gap g which short-circuits the condenser and an oscillatory discharge takes place in the circuit gCL_0. The period of the oscillations depends upon the constants of the circuit.

The equations given above refer only to the permanent conditions, but when the condenser charges and discharges at very frequent intervals, the permanent conditions may not be even reached and the transient phenomena becomes of great importance. We shall give the formulæ for the transient terms in the next chapter.

Another form of transformer which finds considerable application in a high-frequency work is one in which there is only a condenser in the primary circuit as shown in Fig. 52. This arrangement corresponds to a receiving circuit in radiotelegraphy having an untuned secondary.

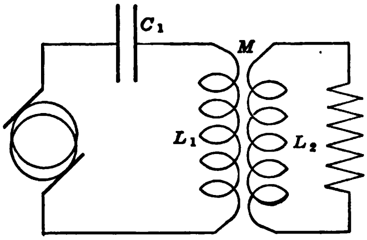

Fig. 52.

If we assume that the incoming signals acting on the antenna are undamped, then we have the following expressions for the amplitudes of the currents in the primary and secondary circuits.

$$I_1 = \frac{E\sqrt{L_2^2\omega^2 + R_2^2}}{\sqrt{\left\{M^2\omega^2 + R_1R_2 - L_2\omega\left(L_1\omega - \frac{1}{C_1\omega}\right)\right\}^2 + \left\{R_2\left(L_1\omega - \frac{1}{C_1\omega}\right) + L_2\omega R_1\right\}^2}}, \tag{94}$$

$$I_2 = \frac{EM\omega}{\sqrt{\left\{M^2\omega^2 + R_1R_2 - L_2\omega\left(L_1\omega - \frac{1}{C_1\omega}\right)\right\}^2 + \left\{R_2\left(L_1\omega - \frac{1}{C_1\omega}\right) + L_2\omega R_1\right\}^2}} \tag{95}$$

If we adjust the inductance and capacity of the primary circuit so as to be in resonance for the frequency of the e.m.f. impressed on the circuit, that is, if we make $L_1\omega - \frac{1}{C_1\omega} = 0$, equations (94) and (95) reduce to

$$\left.\begin{aligned} I_1 &= \frac{E\sqrt{L_2^2\omega^2 + R_2^2}}{\sqrt{\{M^2\omega^2 + R_1R_2\}^2 + L_2^2\omega^2R_1^2}}, \\ I_2 &= \frac{EM\omega}{\sqrt{\{M^2\omega^2 + R_1R_2^2\} + L_2^2\omega^2R_1^2}}. \end{aligned}\right\} \tag{96}$$

The coupling or the mutual inductance which gives the maximum current in the secondary circuit is determined by the equation

$$M^2\omega^2 = R_1\sqrt{R_2^2 + L_2^2\omega^2} = R_1Z_2. \tag{97}$$

For the value of $M\omega$, given by equation (97), we have for the currents in the primary and secondary circuits

$$I_1 = \frac{EZ_2}{R_1\sqrt{2\,(Z_2^2 + R_2Z_2)}}, \quad I_2 = \frac{EM\omega}{R_1\sqrt{2\,(Z_2^2 + R_2Z_2)}}. \tag{98}$$

As an illustration we may use the following constants:

$R_1 = 15$ ohms, $R_2 = 200$ ohms, $L_1 = 1$ mh., $L_2 = 2$ mh.,
$C = 0.001$ mf., $\omega = 10^6$.

By formula (97) the value of M which makes I_2 a maximum is

$$M = 0.1736 \text{ mh.}$$

Introducing these constants in equations (96) we get the following results:

M.	I_1.	I_2.
mh.		
0.1	$6.1 \times 10^{-2} E$	$3.2 \times 10^{-3} E$
0.15	$5.1 \times 10^{-2} E$	$3.84 \times 10^{-3} E$
0.1736	$4.48 \times 10^{-2} E$	$3.89 \times 10^{-3} E$
0.25	$2.80 \times 10^{-2} E$	$3.47 \times 10^{-3} E$
0.50	$0.78 \times 10^{-2} E$	$1.96 \times 10^{-3} E$
1.00	$0.20 \times 10^{-2} E$	$1.00 \times 10^{-3} E$

The Iron-core Transformer.* — In the iron-core transformer we are chiefly concerned with efficiency ratio, regulation and exciting current. It is preferable to consider the transformer from the point of view of the leakage inductance, as this method leads to simpler formulæ than does the treatment from the mutual inductance point of view, and at the same time brings out more clearly the relations between the various factors in the transformer.

The general design formula is

$$E = \pi\sqrt{2}\, n\Phi N \times 10^{-8}, \tag{99}$$

where E is the effective voltage induced in a winding, n the frequency, Φ the maximum total flux linking with the winding and N the number of turns.

Example. — In a 60-cycle transformer whose core is 10 cm. square and which is worked at 10,000 lines per centimeter, the volts per turn are

$$E = \pi\sqrt{2} \times 60 \times 10^2 \times 10{,}000 \times 10^{-8} = 2.7.$$

Let R_1, R_2 = primary and secondary resistance of windings,
X_1, X_2 = primary and secondary leakage reactances of windings,
Z_1, Z_2 = primary and secondary internal impedances,
I_0 = exciting current,
I = load current measured on primary side,

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* William Cramp and C. F. Smith, "Vector and Vector Diagrams Applied to the Alternating Current Circuit," Ch. V.

C. P. Steinmetz, "Theory and Calculation of Alternating Current Phenomena," 3d edition, C. XIV and XV.

V. Karapetoff, "The Electric Circuit."

E_1, E_2 = primary and secondary e.m.f.'s,
$E_{2,0}$ = secondary e.m.f. at no load,
θ = phase angle of connected load, being positive when the secondary voltage leads the secondary current, cos θ being the power factor of the load,
γ = angle by which E_1 leads the exciting current,
α = angle between E_1 and E_2 reversed,
K = ratio of turns, primary to secondary.

Mathematically it is convenient to refer the impedances to the primary. This may be done since an impedance Z_2 in the secondary is equivalent to an impedance K^2Z_2 in the primary. Hence we may write

$$\left.\begin{aligned} R &= R_1 + K^2R_2, \\ X &= X_1 + K^2X_2, \\ Z &= Z_1 + K^2Z^2. \end{aligned}\right\} \qquad (100)$$

In computing the efficiency of ordinary power transformers it is usual to neglect the copper loss due to the exciting current and, unless the contrary is distinctly stated, the efficiency refers to unity power factor load. Hence the formula is simply,

$$\text{per cent efficiency} = \frac{\text{output}}{\text{input}} = \frac{KE_2I}{KE_2I + I^2R + \text{core loss}}. \qquad (101)$$

The core loss is practically equivalent to the power consumption on open secondary.

In an ideally perfect transformer E_1 would be equal to KE_2, but in the actual transformer these differ by the impedance drops in the windings due to the load current and to the exciting current, or

$$\dot{E}_1 - K\dot{E}_2 = (\dot{I}_0 + \dot{I})\,\dot{Z}_1 + K^2\dot{I}\dot{Z}_2 = \dot{I}_0\dot{Z}_1 + \dot{I}\dot{Z}, \qquad (102)$$

where the dotted capitals indicate that the quantities are vector quantities. This is the general equation for every form of transformer.

The leakage reactance of the windings is ordinarily determined by the "short-circuit" test, in which one winding is short-circuited and rated current passed through the windings by applying a voltage to the other winding. From the value of this voltage, the power required and the current, the resistance R and the reactance X, equivalent to a simple impedance, may be readily computed.

In the derivation of the formulæ for regulation, ratio and phase angle, slight approximations are necessary to put the formulæ in usable form, but in all practical cases the formulæ are more accurate than the measurements of the quantities involved can be made experimentally.* The ratio of terminal voltages at any load is

$$\frac{E_1}{E_2} = K + \frac{IR\cos\theta + IX\sin\theta}{E_2} + \frac{(IR\sin\theta - IX\cos\theta)}{2\,KE_2^2} + \frac{I_0\,(R_1\cos\gamma + X_1\sin\gamma)}{E_2}; \qquad (103)$$

for non-inductive load,

$$\frac{E_1}{E_2} = K + \frac{IR}{E_2} + \frac{I^2X^2}{2\,KE_2^2} + \frac{I_0\,(R_1\cos\gamma + X_1\sin\gamma)}{E_2}, \qquad (104)$$

and for no load,

$$\frac{E_1}{E_2} = K + \frac{I_0\,(R_1\cos\gamma + X_1\sin\gamma)}{E_2}. \qquad (105)$$

The regulation at any power factor expressed in per cent is

$$\text{per cent regulation} = 100\left(\frac{E_{2,0} - E_2}{KE_2}\right)$$

$$= 100\left\{\frac{IR\cos\theta + IX\sin\theta}{KE_2} + \frac{(IR\sin\theta - IX\cos\theta)^2}{2\,K^2E_2^2}\right\}; \qquad (106)$$

for non-inductive load,

$$\text{per cent regulation} = 100\left\{\frac{IR}{KE_2} + \frac{I^2X^2}{2\,K^2E_2^2}\right\}. \qquad (107)$$

The phase angle of the transformer is given by

$$\sin\alpha = \frac{I}{E_1}\{X\cos\theta - R\sin\theta\} + \frac{I_0}{E_1}\{X_1\cos\gamma - R_1\sin\gamma\}; \qquad (108)$$

and for the non-inductive load,

$$\sin\alpha = \frac{IX}{E_1} + \frac{I_0}{E_1}(X_1\cos\gamma - R_1\sin\gamma), \qquad (109)$$

and for no load,

$$\sin\alpha_0 = \frac{I_0}{E_1}(X\cos\gamma - R_1\sin\gamma). \qquad (110)$$

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* A derivation of these formulæ showing the magnitude of the approximation involved will appear in Vol. 10 of the *Bulletin of the Bureau of Standards*, by P. G. Agnew and F. B. Silsbee.

The reversed secondary voltage leads the primary voltage for inductive loads and ordinarily at no load, but lags behind it for non-inductive loads.

Example. — Consider a 2 kva., $\frac{1000}{100}$-volt, 60-cycle transformer.

Let $R_1 = 6$, $R_2 = 0.065$, $X = 15$, $I_0 = 0.12$, $\cos\gamma = 0.4$,
$K = 10$, $I = 2$, $R = R_1 + K^2R_2 = 12.5$.

If we assume the reactance drop to be divided equally between primary and secondary, $X_1 = 7.5$.

The computations are correct to 0.01 per cent, although the measurements are not.

Then by (105) the no-load ratio is

$$\frac{E_1}{E_{2,0}} = 10\left\{1 + \frac{0.12\,(6 \times 0.4 + 7.5 \times 0.92)}{1000}\right\} = 10.011.$$

The regulation at 0.6 power factor, lagging current, is by (106)

$$100\left\{\frac{25\times0.6+30\times0.8}{1000} + \frac{(25\times0.8-30\times0.6)^2}{2\times100\times10{,}000}\right\} = 3.9 \text{ per cent.}$$

For leading current the terms in (106) which contain $\sin\theta$ become negative so that at the same power factor leading current the regulation is

$$100\,(-\,0.0090 + 0.0007) = -\,0.83 \text{ per cent.}$$

The minus sign shows that under this condition the secondary voltage is increased by the leading load current.

At non-inductive load the regulation is by (107)

$$100\left\{\frac{25}{1000} - \frac{900}{2{,}000{,}000}\right\} = 2.55 \text{ per cent.}$$

The phase angle at 0.6 power factor, lagging current is by (108)

$$\sin\alpha = \frac{30\times0.6-25\times0.8}{1000} + \frac{0.12\,(7.5\times0.4-6\times0.92)}{1000} = -0.0023,$$

$$\alpha = -\,8',$$

the minus sign meaning, according to the convention adopted, that the reversed secondary voltage leads the primary voltage. From formulæ (109) and (110) we have for non-inductive load,

$$\sin\alpha = \frac{3\,a}{1000} + \frac{0.12\,(7.5 \times 0.4 - 6 \times 0.92)}{1000} = 0.0297, \quad \alpha = 1^\circ\,42',$$

the reversed secondary voltage lagging; and for no load

$$\sin\alpha = \frac{0.12\,(7.5 \times 0.4 - 6 \times 0.92}{1000} = -\,0.0003, \quad \alpha = 1',$$

the reversed secondary voltage leading.

The Current Transformer.*—While the foregoing relations hold also for the current transformer, the conditions of use and the relations desired are so different that a different method of treatment is far more convenient. The resistance and reactance connected to the secondary are kept as low as possible, and the iron is worked at low flux densities, the density being practically proportional to the current. The ratio and phase angle of the current transformer are independent of the resistance and reactance of the primary.

Let I_1, I_2 = primary and secondary currents,
K = ratio of turns, secondary to primary,
M = wattless component of exciting current,
F = core loss component of exciting current,
$\phi = \tan^{-1}\dfrac{\text{total secondary reactance}}{\text{total secondary resistance}}$,
θ = angle between I_1 and I_2.

The ratio of transformation is

$$\frac{I_1}{I_2} = K + \frac{M\sin\phi + F\cos\phi}{I_2}, \tag{111}$$

and the phase angle is given by

$$\tan\theta = \frac{M\cos\phi - F\sin\phi}{KI_2}. \tag{112}$$

If the load connected to the secondary circuit is non-inductive and if, as is usually the case in high-grade transformers, the leakage reactance of the secondary winding may be neglected, equations (111) and (112) become

$$\frac{I_1}{I_2} = K + \frac{F}{I_2}, \tag{113}$$

$$\tan\theta = \frac{M}{KI_2}. \tag{114}$$

Equations (111) to (114) neglect second-order terms, but are sufficiently accurate for all practical work.

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* P. G. Agnew, *Bull. Bureau of Standards*, **7**, 431, 1911. Reprint No. 164.

Example. — A 1000 to 5 ampere current transformer has 198 secondary turns and one primary turn and, when the total resistance and the total reactance of the secondary circuit are 0.4 and 0.5 ohm respectively, the core loss is 0.02 watt, and the exciting current is 8 amperes for a secondary current of 2.5 amperes. Then $K = 198$,

$$\sin\phi = \frac{0.5}{\sqrt{0.16 + 0.25}} = 0.78, \quad \cos\phi = 0.62.$$

The voltage induced in the primary is $\frac{1}{198}\, 2.5\sqrt{0.16+0.25} = 0.0081$.

Hence F, the power component of the exciting current, is

$$0.02 \div 0.0081 = 2.47 \text{ amp.}$$

and M, the wattless component, is $\sqrt{8^2 - (2.47)^2} = 7.6$ amp. Substituting these values of F, M and ϕ in equations (111) and (112), we have for the ratio and phase angle,

$$\frac{I_1}{I_2} = 198 + \frac{7.6 \times 0.78 + 2.5 \times 0.62}{2.5} = 201,$$

and

$$\tan\theta = \frac{7.6 \times 0.62 - 2.5 \times 0.78}{198 \times 2.5} = 0.0056, \quad \theta = 19',$$

the reversed secondary current leading the primary current.

CHAPTER V

TRANSIENT PHENOMENA

If the electrical conditions of a circuit are disturbed in any way, as for instance by a change in the electrical constants of the circuit, or a change in the electromotive force acting on the circuit, a readjustment of the current and potential in the circuit will necessarily follow. The permanent state, however, is not reached instantaneously; it requires an appreciable time interval before electrical equilibrium is again established. The electric phenomena which occur in the time interval before the permanent state is reached again have been properly designated Transient Electric Phenomena. These phenomena are of frequent occurrence and of considerable importance in electrical engineering, particularly whenever high frequencies or high potentials are employed.

It is not within the scope of this book to enter into any discussion of the physical interpretation of these phenomena but, as in the previous chapters, we shall state only the problem, that is, the electrical conditions governing the production of these phenomena, and give the mathematical formulæ expressing the distribution of the current and potential in the circuit. Wherever desirable we shall illustrate the formulæ by numerical examples and curves.

In this chapter we shall limit ourselves to a consideration of circuits of localized inductance and capacity. The transient phenomena which occur in a circuit of distributed inductance and capacity will be discussed in Chapter VI.

INDUCTANCE AND RESISTANCE IN DIRECT CURRENT CIRCUITS*

If we close suddenly a direct current circuit containing resistance and inductance the current does not assume its permanent value instantaneously. It requires a certain interval of time to build up the magnetic field in the inductance coil, and during that interval there is a back e.m.f. produced by the inductance which prevents the current from rising immediately to its permanent value. The value of the current at any instant of time after closing the circuit is given by

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* Ch. P. Steinmetz, "Theory and Calculation of Transient Electric Phenomena and Oscillations," Ch. III.

J. L. LaCour and O. S. Bragstad, "Theorie der Wechselströme," Ch. XXIV.

$$I = I_s\left(1 - \epsilon^{-\frac{R}{L}t}\right), \tag{1}$$

where $I_s = \frac{E}{R}$ is the value of the current in the steady state, and the time is counted from the instant of closing the circuit.

$$\begin{aligned} \text{At } t &= 0, & I &= 0, \\ t &= \infty, & I &= I_s, \text{ permanent value} \end{aligned}$$

$\frac{L}{R}$ is called the time constant of the circuit, and its reciprocal value $\alpha = \frac{R}{L}$ is called the damping factor of the circuit. The time required for the current to rise to n per cent of its steady value is

$$T = \frac{L}{R}\log_\epsilon \frac{1}{1-n} = 2.3\frac{L}{R}\log_{10}\frac{1}{1-\frac{n}{100}}. \tag{2}$$

The smaller the inductance and the larger the resistance the quicker will the current attain its permanent value. In a perfectly non-inductive circuit the current would assume its permanent value instantaneously.

Example. — When $L = 0.1$ henry, $R = 5$ ohms, what is the time required for the current to rise to 90 per cent of its permanent value? By equation (2),

$$T = \frac{0.1}{5}\log_\epsilon \frac{1}{1-0.9} = 0.02 \times 2.3 = 0.046 \text{ sec.}$$

If in the above example we make $L = 1$ henry, then the time required for the current to rise to 90 per cent of its permanent value is 0.46 second, ten times as large as in the former case. In 0.046 second the current would in this case rise only to 21 per cent of its permanent value.

On removing the e.m.f. from a direct current circuit, the current does not vanish immediately. The magnetic field of the inductance produces an e.m.f. which causes a current I to flow in the circuit the value of which is given by the expression

$$I = I_s \epsilon^{-\frac{R}{L}t}, \qquad (3)$$

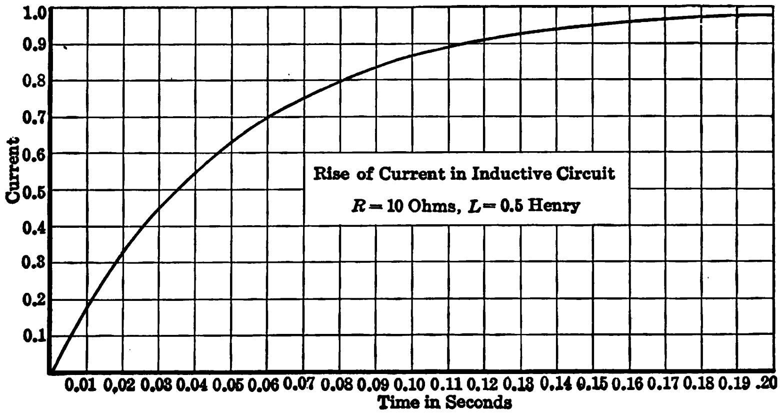

FIG. 53.

where I_s is the steady value of the current before the e.m.f. is removed. The e.m.f. generated by the magnetic field of the inductance coil is

$$E = L\frac{dI}{dt} = -RI_s\epsilon^{-\frac{R}{L}t}. \qquad (4)$$

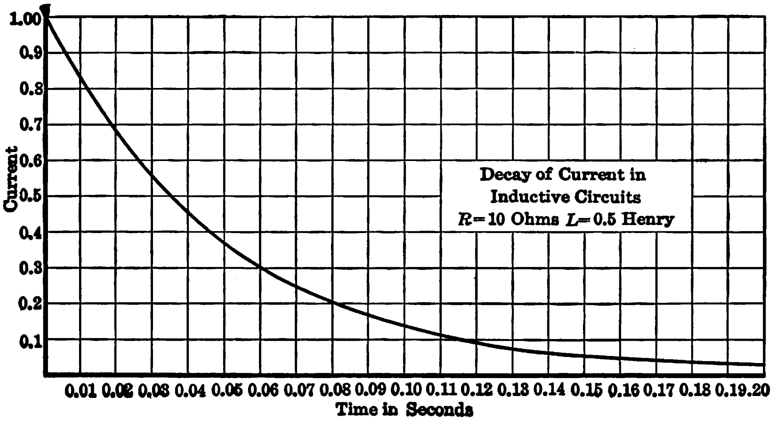

FIG. 54.

In Figs. 53 and 54 two curves are given showing the rise and decay of the currents in an inductive circuit on closing and opening the circuit.

CLOSING AN ALTERNATING CURRENT INDUCTIVE CIRCUIT

Let us assume that at the instant of closing the circuit the e.m.f. of the generator is

$$e = E \sin (\omega t + \psi) = E \sin \psi_{t=0}. \tag{5}$$

The current in the permanent state is

$$I' = I \sin (\omega t + \psi - \phi) = I \sin (\psi - \phi)_{t=0}, \tag{6}$$

where $$I = \frac{E}{\sqrt{\omega^2 L^2 + R^2}}, \quad \tan \phi = \frac{\omega L}{R}.$$

The current, however, will not rise instantly to its permanent value as given by (6) owing to the back e.m.f. generated by the magnetic field of the inductance coil. For a short interval of time after closing the circuit the value of the current will be given by the expression:

$$i = I \sin (\omega t - \psi + \phi) - I \epsilon^{-\frac{R}{L}t} \sin (\psi - \phi). \tag{7}$$

In short-circuiting an alternating current inductive circuit, in which there is no external e.m.f. acting on it, the current gradually dies out in accordance with the expression:

$$i = I \epsilon^{-\frac{R}{L}t} \sin (\psi - \phi). \tag{8}$$

Direct Current Circuits Containing Resistance and Capacity Only, no Inductance. — On closing the circuit the current and potential difference at condenser terminals are given by the expressions

$$I = \frac{E}{R} \epsilon^{-\frac{t}{RC}}, \tag{9}$$

$$e = E \left\{ 1 - \epsilon^{-\frac{t}{RC}} \right\}, \tag{10}$$

where E is the impressed e.m.f. and R and C are the resistance and capacity of the circuit. Equation (9) does not apply for very small values of t, because at $t = 0$, equation (9) gives $I = \frac{E}{R}$, that is, at the moment of closing the current jumped instantly from zero to finite value and this is not possible. To get the value of the current for the very small values of time the inductance of the circuit must be taken into account though it may be extremely small, and in that case we get

$$I = \frac{E}{R}\epsilon^{-\frac{t}{RC}} - \frac{E}{R}\epsilon^{-\frac{R}{L}t}. \tag{11}$$

Example. —

$E = 1000$ volts,
$R = 100$ ohms,
$C = 10$ mf.,
$L = 1$ mh.

t	$\epsilon^{-\frac{t}{RC}}$	$\epsilon^{-\frac{R}{L}t}$	e	I
0	1	1	0	0
10^{-4}	0.990	0.368	10	6.22
10^{-5}	0.904	0	96	9.04
2×10^{-4}	0.818	....	282	8.18
5×10^{-4}	0.606	....	394	6.06
8×10^{-4}	0.449	....	551	4.49
10^{-3}	0.368	....	632	3.68
1.5×19^{-3}	0.223	....	777	2.23
2×10^{-3}	0.135	....	865	1.35
3×10^{-3}	0.050	....	950	0.50
4×10^{-3}	0.018	....	982	0.18
5×10^{-3}	0.007	....	993	0.07

It is to be noticed that the factor $\epsilon^{-\frac{R}{L}t}$ in equation (11) is only operative for an extremely small value time interval, and after that it is entirely negligible and equation (9) may be used.

CIRCUITS CONTAINING RESISTANCE, INDUCTANCE AND CAPACITY IN SERIES*

The problem of determining the current and potential in a circuit containing resistance R, inductance L and capacity C, either on charging the condenser when the circuit is closed, or on discharging the condenser when the circuit is short-circuited, resolves itself into the study of the differential equation

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* W. Thomson, Transient Electric Currents, *Phil. Mag.*, Ser. 4, Vol. 5, p. 393, 1853.

J. A. Fleming, "The Principles of Electric Wave Telegraphy," Ch. I.

C. P. Steinmetz, "Theory and Calculation of Transient Electric Phenomena," Sect. I, Ch. V.

H. Armagnat, "The Theory, Design and Construction of Induction Coils." Translated and edited by O. A. Kenyon.

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=\frac{dE}{dt}.$$

For constant e.m.f. $\frac{dE}{dt}=0$ and the above equation reduces to

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=0. \qquad (12)$$

In working out the solution of equation (13) three different cases may arise depending on the relative magnitudes of L, C and R. They are as follows:

$$\begin{aligned}&\text{(I)}\quad R>2\sqrt{\frac{L}{C}}\text{ (logarithmic case).}\\ &\text{(II)}\quad R=2\sqrt{\frac{L}{C}}\text{ (critical case).}\\ &\text{(III)}\quad R<2\sqrt{\frac{L}{C}}\text{ (trigonometric case).}\end{aligned} \qquad (13)$$

We shall give below the formulæ for the current and potential for all three cases either on charging or discharging.

CASE I. $R>2\sqrt{\frac{L}{C}}$, LOGARITHMIC CASE

On charging, the current in the circuit and the voltage across condenser at any instant of time are as follows:

$$I=\frac{E}{S}\left\{\epsilon^{-\frac{R-S}{2L}t}-\epsilon^{-\frac{R+S}{2L}}\right\}, \qquad (14)$$

$$e=E\left\{1-\frac{1}{2S}\left[(R+S)\epsilon^{-\frac{R-S}{2L}t}-(R-S)\epsilon^{-\frac{R+S}{2L}t}\right]\right\}, \qquad (15)$$

where $$S=\sqrt{R^2-\frac{4L}{C}}.$$

When the condenser is discharging we have for the current and condenser voltage the following equations:

$$I=-\frac{E_0}{S}\left\{\epsilon^{-\frac{R-S}{2L}t}-\epsilon^{-\frac{(R+S)}{2L}t}\right\}, \qquad (16)$$

$$E=\frac{E_0}{2S}\left\{(R+S)\epsilon^{-\frac{R-S}{2L}t}-(R-S)\epsilon^{-\frac{R+S}{2L}t}\right\}, \qquad (17)$$

where E_0 is the condenser voltage before the condenser is beginning to discharge.

The expressions for the currents on charge and discharge of condenser are identical in form but of opposite signs, that is, the currents are exactly the same but in opposite directions. The voltage across condenser increases in one case in exactly the same way as it decreases in the other.

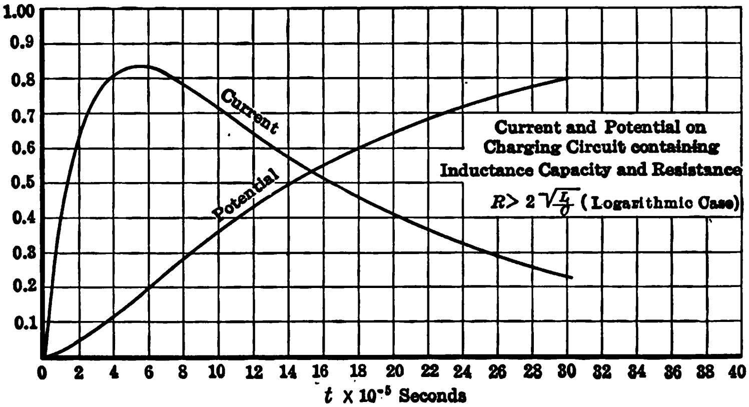

FIG. 55.

An examination of the expressions for the current (14) and (16) shows that the current has zero values for the time $t = 0$ and $t = \infty$; hence, at some instant of time the current has a maximum value. That is, the current is unidirectional, gradually increasing from zero to its maximum value and then decreasing to zero again. The time at which the current is at its maximum value is given by the expression

$$t = \frac{L}{S} \log_e \frac{R+S}{R-S}. \tag{18}$$

Example.—Suppose we have a circuit of the following constants:

$$L = 2\,\text{mh.}, \quad C = 2\,\text{mf.}, \quad R = 100\,\text{ohms}, \quad E = 100\,\text{volts}.$$

$$S = \sqrt{R^2 - \frac{4L}{C}} = \sqrt{10^4 - \frac{8 \times 10^{-3}}{2 \times 10^{-6}}} = 77.$$

$$\frac{R-S}{2L} = 5750, \quad \frac{R+S}{2L} = 44{,}250.$$

The time in which the current will rise to its maximum value is

$$t = \frac{L}{S} \log_e \frac{R+S}{R-S} = \frac{2 \times 10^{-3}}{77} \log_e \frac{177}{23} = 5.3 \times 10^{-5} \text{ sec.}$$

CASE II. $R^2 = \frac{4L}{C}$, CRITICAL CASE

The current and voltage across condenser on charging are

$$I = \frac{t}{L} e^{-\frac{R}{2L}t}, \tag{19}$$

$$e = E\left\{1 - \left(1 + \frac{Rt}{L}\right)\epsilon^{-\frac{R}{2L}t}\right\}. \tag{20}$$

The current and condenser voltage on discharging are

$$\left.\begin{aligned} I' &= -\frac{t}{L} E_0 \epsilon^{-\frac{R}{L}t}, \\ e' &= \left(1 + \frac{Rt}{L}\right) E_0 \epsilon^{-\frac{R}{2L}t}. \end{aligned}\right\} \tag{21}$$

CASE III. $R < 2\sqrt{\frac{L}{C}}$, TRIGONOMETRIC CASE

The current and voltage across condenser terminals on charging are

$$I = \frac{E}{L\beta} \epsilon^{-\alpha t} \sin \beta t, \tag{22}$$

$$e = E\left\{1 - \epsilon^{-\alpha t}\left(\cos \beta t + \frac{\alpha}{\beta} \sin \beta t\right)\right\}, \tag{23}$$

where $\alpha = \frac{R}{2L}, \quad \beta = \sqrt{\frac{1}{LC} - \alpha^2}.$

For condenser discharge we have for the current and condenser voltage the following equations:

$$I = -\frac{E_0}{L\beta} e^{-\alpha t} \sin \beta t, \tag{24}$$

$$e_1 = E_0 \epsilon^{-\alpha t}\left(\cos \beta t + \frac{\alpha}{\beta} \sin \beta t\right). \tag{25}$$

In this case also the currents have the same numerical values on charge and discharge but are of opposite signs, and for the condenser potentials the increase in one case is exactly equal to the decrease in the other case.

Equations (22) to (25) represent damped periodic curves, that is, the current and potential are oscillatory and the amplitudes are continuously diminishing. The maximum value of the current decreases in geometric progression as the time increases in arithmetic progression.

The frequency of oscillation is

$$n = \frac{\beta}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \alpha^2}. \tag{26}$$

The frequency decreases as the resistance in the circuit is increased and becomes zero, ceases to be oscillatory, when $\frac{1}{LC} = \frac{R^2}{4L^2}$ (critical case). When the resistance is small so that R^2 can be neglected in comparison with $\frac{4L}{C}$ we have

$$n = \frac{1}{2\pi\sqrt{LC}}. \tag{27}$$

The quantity $\frac{R}{2L}$ denoted by α is called the damping factor. If the resistance is small $\frac{\alpha}{\beta}$ is small compared with unity and equations (24) and (25) reduce to

$$I = - E_0 C \beta \epsilon^{-\alpha t} \sin \beta t, \tag{28}$$

$$e_1 = E_0 \epsilon^{-\alpha t} \cos \beta t; \tag{29}$$

or we may put

$$I = E_0 \sqrt{\frac{C}{L}} \epsilon^{-\alpha t} \cos \beta t. \tag{30}$$

The quotient of the current to the condenser voltage is $\sqrt{\frac{C}{L}}$. If the circuit is entirely free from resistance, the oscillations persist indefinitely, and they are generally designated as undamped or free oscillations.

The ratio of the amplitudes of successive half waves is

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = \text{etc.} = \epsilon^{-\frac{\alpha T}{2}}, \tag{31}$$

where T is the duration of a complete period. The term $\frac{\alpha T}{2} = \frac{\alpha}{2n} = \frac{R}{4Ln}$ is denoted by δ and is called the logarithmic decrement of the oscillation per half period.

Hence,

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = \text{etc.} = \epsilon^{-\delta},$$

and

$$\delta = \log_\epsilon \frac{I_1}{I_2} = \log_\epsilon \frac{I_2}{I_3}, \text{ etc.} \tag{32}$$

Since $n = \frac{1}{2\pi\sqrt{LC}}$, approximately, the expression for the logarithmic decrement may be put in the form

$$\delta = \frac{R}{4Ln} = \frac{\pi R}{2}\sqrt{\frac{C}{L}}. \tag{33}$$

As an illustration we may take the following constants:

$L = 1$ mh., $C = 0.001$ mf., $R = 100$ ohms, $E = 1000$ volts.

$$\frac{1}{LC} = 10^{12}, \quad \frac{R}{2L} = 5 \times 10^4.$$

$$\frac{R^2}{4L^2} \text{ is negligible compared with } \frac{1}{LC}.$$

$$\beta = \sqrt{\frac{1}{LC}} = 10^6, \quad \alpha = 5 \times 10^4, \quad E_0\sqrt{\frac{C}{L}} = 1.$$

The ratio of two successive half waves is

$$\frac{I_1}{I_2} = \frac{0.92}{0.79} = \frac{0.79}{0.68} = \text{etc.} = 1.164,$$

$$\delta = \log_\epsilon 1.164 = 0.152.$$

By (33) we get $\delta = \frac{\pi \times 100}{2}\sqrt{\frac{10^{-9}}{10^{-3}}} = 0.157.$

The small difference in the values of δ arises on account of carrying only to two decimal places the values of $\epsilon^{-\alpha t}$ in calculating the values of I_1, I_2, etc.

If we introduce a hot-wire ammeter in the oscillating circuit, the current indicated is the effective value of the square root of the mean square of the discharge current. Generally the time interval between two successive discharges is sufficiently long to allow complete discharge of the condenser. That is, the current is practically zero before another discharge takes place. To get the effective value of the current we may therefore integrate equation (24) from zero to infinity and divide by the time interval between two successive discharges. The square root of the value thus obtained will be the effective value of the discharge current.

If we denote by N the number of discharges per second, we have

$$I_{\text{eff}} = E_0\sqrt{\frac{CN}{2R}}. \tag{34}$$

If the condenser is charged N times a second to a potential E_0 the power supplied is

$$P = \frac{NCE_0^2}{2}. \tag{35}$$

We can also get an expression for the effective value of the current in terms of the first maximum value of the oscillation, as follows:

$$I_{\text{eff}} = \sqrt{\frac{NI_1^2\epsilon^\delta}{8n\delta} \times \frac{1}{1+\left(\frac{\delta}{\pi}\right)^2}}, \tag{36}$$

where δ is the logarithmic decrement, n is the frequency of oscillations and N is the number of discharges. For oscillatory currents of high frequency such as are generally employed in wireless telegraphy $\frac{\delta}{\pi}$ is generally a small quantity not greater than 0.1, and often much smaller than that, hence we may neglect $\left(\frac{\delta}{\pi}\right)^2$ in comparison with unity, and equation (36) reduces to

$$I_{\text{eff}} = \sqrt{\frac{NI_1^2\epsilon^\delta}{8n\delta}} = I_1\sqrt{\frac{N\epsilon^\delta}{4\alpha}}. \tag{37}$$

Example. — Let us take the same data as in the illustration given on page 180.

$E_0 = 100$ volts, $C = 10^{-3}$ mf., $L = 1$ mh., $R = 100$ ohms, $I_1 = 0.92$, $n = \frac{10^6}{2\pi}$, $\delta = 0.157$.

Assuming $N = 1000$ we have by (34)

$$I_{\text{eff}} = 1000\sqrt{\frac{10^3 \times 10^{-9}}{200}} = 0.071 \text{ amp.}$$

By (36) we have

$$\epsilon^{\delta} = 1.17.$$

$$I_{\text{eff}}^2 = \frac{10^3\,(0.92)^2 \times 1.17}{\dfrac{8 \times 10^6}{2\pi} \times 0.157} \times \frac{1}{1 + \left(\dfrac{0.157}{\pi}\right)^2} = \frac{5.04 \times 10^{-3}}{1 + \left(\dfrac{0.157}{\pi}\right)^2}$$

$$= 5.04 \times 10^{-3}, \text{ approximately.}$$

$$I_{\text{eff}} = 0.071 \text{ amp.},$$

which is in agreement with the value obtained by formulæ (34). It is also to be noted that the term $\left(\frac{\delta}{\pi}\right)^2$ is very small compared with unity, and formula (37) would have given the same results.

Alternating Current Circuits Containing Resistance, Inductance and Capacity in Series. — When an alternating e.m.f. is introduced into a circuit having a resistance R, inductance L and capacity C the current and condenser voltage do not attain their permanent values instantly. It requires a certain time interval, which is generally very small in all practical cases, before the current and voltage across condenser have built up to their maximum value.

The current and condenser voltage may be expressed as the difference of two quantities,

$$\left.\begin{aligned} I &= I_s - I_v, \\ E &= E_s - E_v, \end{aligned}\right\} \qquad (38)$$

where the subscript s denotes the steady or permanent values and the subscript v denotes only the variable values of the current and voltage which are present during the transition period. I_v and E_v diminish in accordance with an exponential law with respect to time and consequently disappear very rapidly. Formulæ for I_s and E_s have been given in Chapter IV, hence we shall give here only formulæ for I_v and E_v.

As in the previous section we have to distinguish here also three distinct cases depending on the relative magnitudes of the electrical constants L, C and R.

Let us consider first the trigonometric or oscillatory case

$$R < 2\sqrt{\frac{L}{C}}.$$

We shall assume that at the instant of closing the circuit the steady values of the voltage and current are

$$\left.\begin{aligned} E_s &= E_{\max}\sin(\omega t+\psi) = E_{\max}\sin\psi_{t=0}, \\ I_s &= I_{\max}\sin(\omega t+\psi-\phi) = I_{\max}\sin(\psi-\phi), \end{aligned}\right\} \tag{39}$$

where

$$I_{\max} = \frac{E_{\max}}{\sqrt{\left(\omega L - \dfrac{1}{\omega C}\right)^2 + R^2}}, \quad \tan\phi = \frac{\omega L - \dfrac{1}{\omega C}}{R}.$$

For the limiting conditions given by (39) we have the following formulæ for I_v and E_v:

$$I_v = \frac{\epsilon^{-\alpha t}}{\beta L}\left\{E_{\max}\cos(\psi-\phi)\sin\beta t + I_{\max}\sqrt{\frac{L}{C}}\sin(\psi-\phi)\sin(\beta t-\gamma)\right\}, \tag{40}$$

$$E_v = -\epsilon^{-\alpha t}\left\{\frac{E_{\max}}{\beta\sqrt{LC}}\cos(\psi-\phi)\sin(\beta t+\gamma) + \frac{I_{\max}\sin(\psi-\phi)}{\beta C}\sin\beta t\right\}, \tag{41}$$

where

$$\alpha = \frac{R}{2L}, \quad \beta = \sqrt{\frac{1}{LC} - \alpha^2}, \quad \gamma = \tan^{-1}\frac{\beta}{\alpha}.$$

Any change in the electrical constants of the circuit will also introduce a transient term. The current and potential cannot pass over instantly from one permanent value to another, it requires a certain time interval for the readjustment. During the transition period the current and potential can also be expressed by formulæ (38), but in this case the values of I_v and E_v are as follows:

$$I_v = \frac{\epsilon^{-\alpha t}}{\beta L}\left\{\Delta E\sin\beta t + \Delta I\sqrt{\frac{L}{C}}\sin(\beta t-\gamma)\right\}, \tag{42}$$

$$E_v = \epsilon^{-\alpha t}\left\{\frac{\Delta E}{\beta\sqrt{LC}}\sin(\beta t-\gamma) + \frac{\Delta I}{\beta C}\sin\beta t\right\}, \tag{43}$$

where

$$\left.\begin{aligned} \Delta E &= (E_{2,s} - E_{1,s})_{(t=0)}, \\ \Delta I &= (I_{2,s} - I_{1,s})_{(t=0)}. \end{aligned}\right\} \tag{44}$$

Equations (44) give the variation in the permanent values of the current and condenser potential in the circuit at the time $t = 0$.

For the critical case $R = 2\sqrt{\frac{L}{C}}$ the value of I_v is given by the formula

$$I_v = \epsilon^{-\alpha t}\left\{E_{c,\max}\frac{t}{L}\cos(\psi - \phi) - I_{\max}\left(1 - \frac{\alpha}{t}\right)\sin(\psi - \phi)\right\}. \quad (45)$$

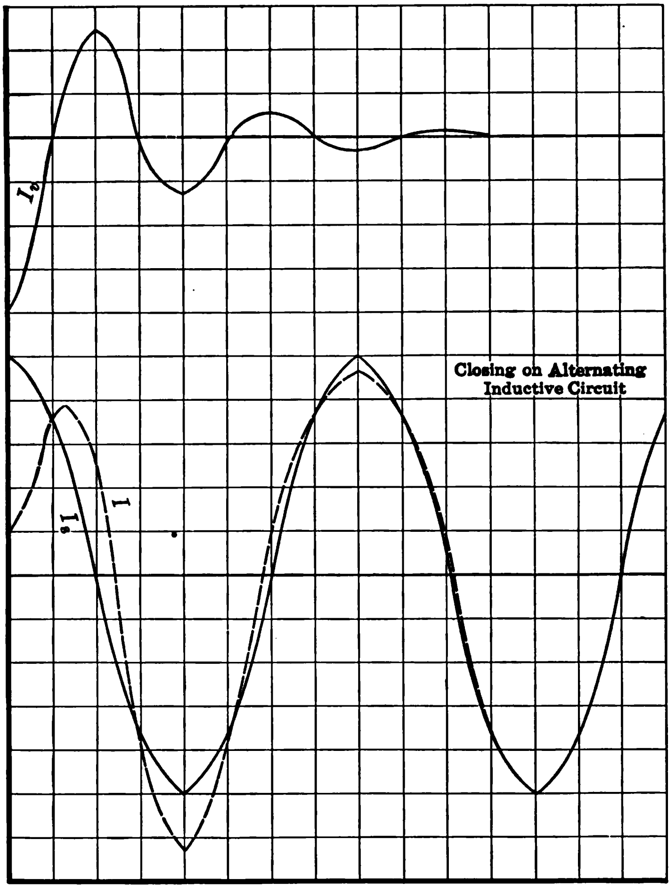

Fig. 56.

When the resistance is still further increased, so that $R > 2\sqrt{\frac{L}{C}}$, we have the logarithmic case, and the value of I_v is

$$I_v = \frac{\epsilon^{-\alpha t}}{2L\beta}\left\{\left[E_{c,\max}\cos(\psi-\phi)+\left(\frac{R}{2}-L\beta\right)I_{\max}\sin(\psi-\phi)\right]\epsilon^{\beta t} - \left[E_{c,\max}\cos(\psi-\phi)+\left(\frac{R}{2}+L\beta\right)I_{\max}\sin(\psi-\phi)\right]\epsilon^{-\beta t}\right\}. \quad (46)$$

As in the trigonometric case we have here also for any change in load condition a transition period during which we have, besides the term representing the permanent value of the current, another term which is effective for a short period only, the transient term. The value of the transient term which we designated throughout by I_v is as follows:

$$I_v = \frac{\epsilon^{-\alpha t}}{2L\beta}\left\{\left[\Delta E+\left(\frac{R}{2}-L\beta\right)\Delta I\right]\epsilon^{\beta t} - \left[\Delta E+\left(\frac{R}{2}+L\beta\right)\Delta I\right]\epsilon^{-\beta t}\right\}, \quad (47)$$

where

$$\left.\begin{aligned}\Delta I &= (I_{2,s} - I_{1,s})_{(t=0)},\\ \Delta E &= (E_{2,c} - E_{1,c})_{(t=0)}\end{aligned}\right\} \quad (48)$$

represent the differences in I and E between the steady values before and after the change.

The values of α and β in equations (46) and (47) are given by

$$\alpha = \frac{R}{2L}, \quad \beta = \sqrt{\alpha^2 - \frac{1}{LC}}.$$

Divided Circuits.*—We shall first consider the simplest case, the circuits having only inductance and resistance.

Suppose we have a continuous current flowing through the circuits and either the main circuit is short-circuited, or a variation in the resistance of any of the branches is made. The problem is to determine the values of the transient terms of the currents in the branches.

Let E = e.m.f.

L_0, L_1, L_2 = inductances of main and branch circuits respectively.

R_0, R_1, R_2 = resistances of main and branch circuits respectively.

The permanent values of the currents in branches 1 and 2 are

$$I_1 = \frac{ER_2}{R^2} \quad \text{and} \quad I_2 = \frac{ER_1}{R^2}, \quad (49)$$

where

$$R^2 = R_0R_2 + R_1R_0 + R_1R_2.$$

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* See Ch. P. Steinmetz, "Theory and Calculation of Transient Electric Phenomena and Oscillations," Section I, Ch. IX.

Now let us suppose that there is a change in resistance in branch 1 or 2. The currents in the two branches are as follows:

$$I_1 = I_1'' + \left\{ \frac{K_2 (I_1' - I_1'') + I_2' - I_2''}{K_2 - K_1} \right\} \epsilon^{\lambda_1 t} - \left\{ \frac{K_1 (I_1' - I_1'') + I_2' - I_2''}{K_2 - K_1} \right\} \epsilon^{\lambda_2 t}, \tag{50}$$

$$I_2 = I_2'' - \frac{K_1 \{K_2 (I_1' - I_1'') + I_2' - I_2''\}}{K_2 - K_1} \epsilon^{\lambda_1 t} + \frac{K_2 \{K_1 (I_1' - I_1'') + I_2' - I_2''\}}{K_2 - K_1} \epsilon^{\lambda_2 t}, \tag{51}$$

where I_1' and I_2' are the permanent values of the currents in the two branches before the change in the resistance, and I_1'' and I_2'' are the permanent values of the currents after the change.

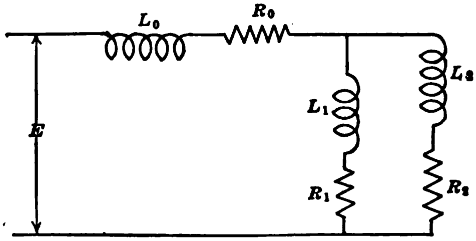

FIG. 57.

$$L^2 = L_0 L_2 + L_0 L_1 + L_1 L_2,$$
$$R^2 = R_0 R_2 + R_0 R_1 + R_1 R_2,$$
$$S^2 = R_2 (L_0 + L_1) + R_0 (L_1 + L_2) + R_1 (L_2 + L_0),$$
$$\lambda_1 = \frac{-S^2 + \sqrt{S^4 - 4 L^2 R^2}}{2 L^2}, \quad \lambda_2 = \frac{-S^2 - \sqrt{S^4 - 4 L^2 R^2}}{2 L^2},$$
$$-K_1 = \frac{(L_0 + L_1)\lambda_1 + (R_0 + R_1)}{L_0 \lambda_1 + R_0}, \quad -K_2 = \frac{(L_0 + L_2) + (R_0 + R_1)}{L_0 \lambda_2 + R_0}.$$

As an illustration let us consider the following example:

$L_0 = 5$ henrys, $L_1 = 2.5$ henrys, $L_2 = 10$ henrys,
$R_0 = 10$ ohms, $R_1 = 2$ ohms, $R_2 = 25$ ohms,

and we shall assume that the value of R_2 is suddenly changed to 15 ohms.

The permanent values of the currents before and after the change in the resistance are

$$I_1' = 7.81 \text{ amp.}, \qquad I_2' = 0.625 \text{ amp.},$$
$$I_1'' = 7.5 \text{ amp.}, \qquad I_2'' = 1.00 \text{ amp.}$$

We also have

$$L^2 = 87.5, \qquad R^2 = 200, \qquad S^2 = 267.5.$$
$$\lambda_1 = -1.42, \qquad \lambda_2 = -1.63,$$
$$K_1 = 0.47, \qquad K_2 = -0.12,$$

Introducing these values in equations (50) and (51) we get

$$I_1 = 7.5 + 0.70\,\epsilon^{-1.42t} - 0.39\,\epsilon^{-1.63t},$$
$$I_2 = 1.00 - 0.33\,\epsilon^{-1.42t} - 0.045\,\epsilon^{-1.63t}.$$

For $t = 0$

$$\left.\begin{aligned} I_1 &= 7.81 \\ I_2 &= 0.625 \end{aligned}\right\} \text{ which are the permanent values of the current before the change in resistance.}$$

For $t = \infty$

$$\left.\begin{aligned} I_1 &= 7.5 \text{ amp.} \\ I_2 &= 1.00 \text{ amp.} \end{aligned}\right\} \text{ which are the permanent values of the current after the change in resistance.}$$

If the circuit is short-circuited we must put in equations (50) and (51) I_1'' and I_2'' separately equal to zero, and the equations reduce to

$$I_1 = \frac{K_2 I_1' + I_2'}{K_2 - K_1}\,\epsilon^{\lambda_1 t} - \frac{K_1 I_1' + I_2'}{K_2 - K_1}\,\epsilon^{\lambda_2 t}, \tag{52}$$

$$I_2 = -\frac{K_1 K_2 I_1' + K_1 I_2'}{K_1 - K_2}\,\epsilon^{\lambda_1 t} + \frac{K_1 K_2 I_2' + K_2 I_2'}{K_1 - K_2}\,\epsilon^{\lambda_2 t}. \tag{53}$$

Using the same constants as in the above example, we get

$$I_1 = 0.53\,\epsilon^{-1.42t} + 7.28\,\epsilon^{-1.63t},$$
$$I_2 = -0.25\,\epsilon^{-1.42t} + 0.87\,\epsilon^{-1.63t}.$$

Transient Currents in Transmission Lines.* — The capacity of the transmission line is represented by a condenser shunted across the middle of the line. We shall give here formulæ for the transient currents and potentials which may arise either when the load is short-circuited or the generator is short-circuited.

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* See J. L. LaCour and O. S. Bragstad, "Theorie der Wechselströme," Ch. XXIV.

Consider first the case of the transient phenomena which occur in transmission line when the load is short-circuited, as shown diagrammatically in Fig. 58.

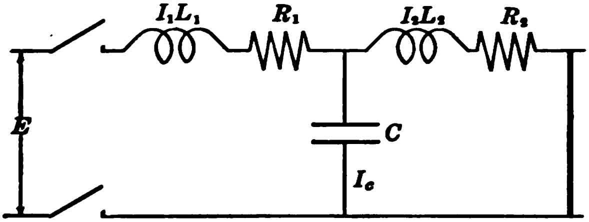

Fig. 58.

Let $I_{1,s,0}$, $I_{2,s,0}$, $I_{c,s,0}$, $E_{c,s,0}$ represent the permanent values of the currents in the different branches and the condenser potential at the instant that the line is short-circuited at the receiving end. Then we have the following formulæ for the transient values of the currents and potential.

$$\left.\begin{aligned} I_{1,v} = \epsilon^{-\alpha t}&\left\{\frac{E_{c,s,0}}{\beta L_2}\sin\beta t + \sqrt{1+\frac{\alpha^2}{\beta^2}}\,\frac{I_{c,s,0}}{2}\sin(\beta t-\phi)\right\} \\ &-\left(I_{2,s,0}+\frac{I_{c,s,0}}{2}\right)\epsilon^{-\delta t}, \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} I_{2,v} = \epsilon^{-\alpha t}&\left\{-\frac{E_{c,s,0}}{\beta L_2}\sin\beta t + \sqrt{1+\frac{\alpha^2}{\beta^2}}\,\frac{I_{c,s,0}}{2}\sin(\beta t-\phi)\right\} \\ &-\left(I_{2,s,0}+\frac{I_{c,s,0}}{2}\right)\epsilon^{-\delta t}, \end{aligned}\right\} \tag{55}$$

$$E_{c,v} = \epsilon^{-\alpha t}\left\{-\frac{I_{c,s,0}}{\beta C}\sin\beta t - \sqrt{1+\frac{\alpha^2}{\beta^2}}\,E_{c,s,0}\sin(\beta t+\phi)\right\}, \tag{56}$$

where

$$\alpha = \frac{R_1}{2\,L_1} = \frac{R_2}{2\,L_2},$$

$$\beta = \sqrt{\frac{2}{L_1C}-\alpha^2} = \sqrt{\frac{2}{L_2C}-\alpha^2},$$

$$\delta = \frac{R_1}{L_1} = \frac{R_2}{L_2} = 2\,\alpha \quad \text{and} \quad \phi = \tan^{-1}\frac{\beta}{\alpha}.$$

The values of α and β given above have been obtained on the assumption that $\frac{R_1}{L_1} = \frac{R_2}{L_2}$.

An examination of equation (56) makes it appear evident that no great rise in potential can occur on the line.

As an illustration consider the case of a single-phase line 100 miles long, made up of No. 000 B. & S. with wires 96 inches interaxial distance. The resistance per mile $R = 0.33$ ohm.

Inductance per mile $L = 2.06$ mh.
Capacity per mile $C = 0.0145$ mf.

The total resistance, inductance and capacity of the line are 33 ohms, 206 mh. and 1.45 mf. respectively.

We have therefore

$$R_1 = R_2 = 16.5 \text{ ohms.}$$

$$L_1 = L_2 = 103 \text{ mh.} \quad \alpha = \frac{R_1}{2L_1} = 80.1,$$

$$\beta = \sqrt{\frac{2}{103 \times 10^{-3} \times 1.45 \times 10^{-6}} - (80.1)^2} = 36.6 \times 10^2,$$

$$\beta C = 0.005, \quad \sqrt{1 + \frac{\alpha^2}{\beta^2}} = \sqrt{1.004} = 1, \text{ approx.}$$

$$E_{c,v} = -\epsilon^{-80t}\left\{\frac{I_{c,s,0}}{0.005}\sin\beta t + E_{c,s,0}\sin(\beta t + \phi)\right\}$$

$$= -\epsilon^{-80t}\{200\,I_{c,s,0}\sin\beta t + E_{c,s,0}\sin(\beta t + \phi)\}.$$

If $I_{c,s,0}$ is numerically equal to one hundredth of $E_{c,s,0}$, that is, $I_{c,s,0} = \frac{E_{c,s,0}}{100}$, the rise in potential is only about three times the normal value.

For the case where the generator is short-circuited we may neglect L_2 compared with R_2, and in that case the transient current in the branches and the potential is given by the following expressions:

$$I_{2,v} = \epsilon^{-\alpha t}\left\{\frac{I_{c,s,0}}{\beta C R_2}\sin\beta t + I_{2,s,0}\sqrt{1 + \frac{\alpha^2}{\beta^2}}\sin(\beta t + \phi)\right\}, \tag{57}$$

$$I_{1,v} = \epsilon^{-\alpha t}\left\{\frac{-I_{1,s,0,}R_1 + E_{c,s,0}}{\beta L_1}\sin\beta t + I_{1,s,0}\sqrt{1 + \frac{\alpha^2}{\beta^2}}\sin(\beta t + \phi)\right\}, \tag{58}$$

$$E_{c,v} = \epsilon^{-\alpha t}\left\{\frac{I_{c,s,0}}{\beta C}\sin\beta t + E_{c,s,0}\sqrt{1 + \frac{\alpha^2}{\beta^2}}\sin(\beta t + \phi)\right\}, \tag{59}$$

where

$$\alpha = \frac{1}{2\,R_2C} + \frac{R_1}{2\,L_1},$$

$$\beta = \sqrt{\frac{R_1 + R_2}{R_2L_1C} - \alpha^2},$$

$$\phi = \tan^{-1}\frac{\beta}{\alpha}.$$

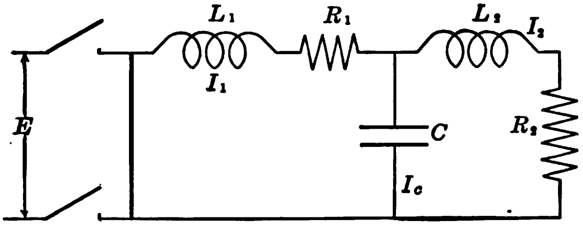

Fig. 59.

If a direct e.m.f. is acting on the line, then $E_{c,s,0} = E$ and $I_{c,s,0} = 0$, where E is the e.m.f. impressed on the line. Equation (59) reduces to

$$\left.\begin{aligned} E_{c,v} &= \epsilon^{-\alpha t}\,E\sqrt{1+\frac{\alpha^2}{\beta^2}}\,\sin(\beta t+\phi), \\ \frac{E_{c,v,\,\max}}{E} &= \sqrt{1+\frac{\alpha^2}{\beta^2}}. \end{aligned}\right\} \qquad (60)$$

The increase in potential is proportional to

$$\sqrt{1+\frac{\alpha^2}{\beta^2}} = \frac{1}{\beta\sqrt{L_1C}}\sqrt{\frac{R_1+R_2}{R_2}}.$$

This ratio is seldom much greater than unity and increases as α increases.

Mutual Inductance.* — If we have two circuits connected inductively and impress continuous e.m.f.'s E_1 and E_2 on circuits I and II respectively the currents in the circuits are

$$I_1' = \frac{E_1}{R_1} \quad \text{and} \quad I_2' = \frac{E_2}{R_2}. \qquad (61)$$

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* See Ch. P. Steinmetz, "Theory and Calculation of Transient Electric Phenomena," Section I, Ch. X.

Einführung in die Maxwellsch Theorie der Elektrizität von Dr. A. Föppl herausgegeben von Dr. M. Abraham, Part III, Ch. II.

If there is a sudden change in load conditions, say R_2 is changed to R_2', the permanent values of the currents in the circuits after the change are

$$I_1'' = \frac{E_1}{R_1} \quad \text{and} \quad I_2'' = \frac{E_2}{R_2'}. \tag{62}$$

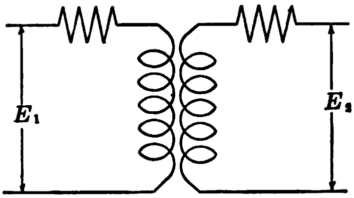

FIG. 60.

The change, however, from one current value to another due to a change in circuit conditions does not occur instantly. The currents rise gradually to the changed permanent values; the interval of time required before the permanent conditions are established again depends upon the electric constants of the circuits.

The expression for the currents in the two circuits at any time t are as follows:

$$I_1 = I_1' + \frac{\Delta I_2}{m_2 - m_1}\epsilon^{\lambda_1 t} - \frac{\Delta I_2}{m_2 - m_1}\epsilon^{\lambda_2 t}, \tag{63}$$

$$I_2 = I_2'' - \frac{m_1 \Delta I_2}{m_2 - m_1}\epsilon^{\lambda_1 t} + \frac{m_2 \Delta I_2}{m_2 - m_1}\epsilon^{\lambda_2 t}, \tag{64}$$

where

$$\Delta I_2 = I_2' - I_2'',$$

$$m_1 = \frac{L_1\lambda_1 + R_1}{m\lambda_1}, \quad m_2 = \frac{L_1\lambda_2 + R_1}{m\lambda_2},$$

$$\left.\begin{aligned} \lambda_1 &= \frac{-(\alpha_1 + \alpha_2) + \sqrt{(\alpha_2 - \alpha_1)^2 + 4\,\alpha_1\alpha_2 K^2}}{1 - K^2}, \\ \lambda_2 &= \frac{-(\alpha_1 + \alpha_2) - \sqrt{(\alpha_2 - \alpha_1)^2 + 4\,\alpha_1\alpha_2 K^2}}{1 - K^2}, \\ \alpha_1 &= \frac{R_1}{2\,L_1}, \quad \alpha_2 = \frac{R_2}{2\,L_2}, \end{aligned}\right\} \tag{65}$$

$$K^2 = \frac{M^2}{L_1 L_2}.$$

If there is a change in the resistance of circuit I from R_1 to R_1' the permanent values of the currents in the circuits after the change are

$$I_1'' = \frac{E}{R_1'} \quad \text{and} \quad I_2'' = \frac{E_2}{R_2}. \tag{66}$$

The currents in the circuits before the permanent values are established are given by the expressions.

$$I_1 = I_1'' - \frac{\Delta I_1 m_2}{m_1 - m_2}\epsilon^{\lambda_1 t} + \frac{\Delta I_1 m_1}{m_1 - m_2}\epsilon^{\lambda_2 t}, \tag{67}$$

$$I_2 = I_2'' + \frac{\Delta I_1 m_1 m_2}{m_1 - m_2}\epsilon^{\lambda_1 t} - \frac{\Delta I_1 m_1 m_2}{m_1 - m_1}\epsilon^{\lambda_2 t}, \tag{68}$$

where $\Delta I_1 = I_1' - I_1''$ and $m_1, m_2, \lambda_1, \lambda_2$ have the same significance as given in equation (65).

If one of the circuits is suddenly opened, we shall merely have to put either $I_2'' = 0$ in equations (63) and (64) or $I_1'' = 0$ in equation (67) and (68) depending upon whether we open circuit II or circuit I.

Example. —

$L_1 = 0.1$ henry, $R_1 = 10$ ohms, $L_2 = 0.2$ henry,
$R_2 = 25$ ohms, $M = 0.1$ henry, $E_1 = 100$ volts,
$E_2 = 200$ volts.

Now suppose we suddenly change R_2 to 40 ohms. We have

$$I_1' = \frac{100}{10} = 10 \text{ amp.}, \quad I_2' = \frac{200}{25} = 8 \text{ amp.}, \quad I_2'' = \frac{200}{40} = 5 \text{ amp.},$$

$$\alpha_1 = \frac{10}{0.2} = 50, \quad \alpha_2 = \frac{25}{0.4} = 62.5, \quad K^2 = \frac{0.01}{0.02} = 0.5,$$

$$\lambda_1 = \frac{-(50 + 62.5) + \sqrt{(62.5 - 50)^2 + 4 \times 50 \times 62.5 \times 0.5}}{1 - 0.5} = -65,$$

$$\lambda_2 = \frac{-(50 + 62.5) - \sqrt{(62.5 - 50)^2 + 4 \times 50 \times 62.5 \times 0.5}}{1 - 0.5} = -385,$$

$$m_1 = \frac{-0.1 \times 65 + 10}{-0.1 \times 65} = -0.54, \quad m_2 = \frac{-0.1 \times 385 + 10}{-0.1 \times 385} = 0.74,$$

$$\Delta I_2 = 8 - 5 = 3;$$

hence,

$$I_1 = 10 + \frac{3}{128}\epsilon^{-65t} - \frac{3}{128}\epsilon^{-385t},$$

$$I_2 = 5 + 1.26\,\epsilon^{-65t} + 1.74\epsilon^{-385t}.$$

For $t = 0$

$$\left.\begin{aligned} I_1 &= 10 \text{ amp.} \\ I_2 &= 8 \text{ amp.} \end{aligned}\right\} \text{permanent values before change in resistance of circuit II.}$$

For $t = \infty$

$$\left.\begin{aligned} I_1 &= 10 \text{ amp.} \\ I_2 &= 5 \text{ amp.} \end{aligned}\right\} \text{permanent values after change in resistance of circuit II.}$$

If in the above example we had changed R_1 from 10 to 20 ohms and kept R_2 constant, we should have had

$$I_1' = \frac{100}{10} = 10 \text{ amp.,} \quad I_1'' = \frac{100}{20} = 5 \text{ amp.,} \quad I_2' = \frac{200}{25} = 8 \text{ amp.}$$

m_1, m_2, λ_1, λ_2 have the same values as above.

By (67) and (68) we get

$$I_1 = 5 + 2.9\,\epsilon^{-65\,t} + 2.1\,\epsilon^{-385\,t},$$
$$I_2 = 8 + 1.56\,\epsilon^{-65\,t} - 1.56\,\epsilon^{-385\,t}.$$

For $t = 0$

$$\left.\begin{aligned} I_1' &= 10 \text{ amp.} \\ I_2' &= 8 \text{ amp.} \end{aligned}\right\} \text{permanent values before the change in resistance of circuit I.}$$

For $t = \infty$

$$\left.\begin{aligned} I_1'' &= 5 \text{ amp.} \\ I_2'' &= 8 \text{ amp.} \end{aligned}\right\} \text{permanent values after the change in resistance of circuit I.}$$

Let us now consider the case where one circuit has no e.m.f. acting on it and an e.m.f. E_1 is introduced in the other circuit. Suppose $E_2 = 0$. There is no current in either circuit before the e.m.f. is introduced in circuit I; that is, we have

$$I_1' = 0, \quad I_2' = 0, \quad I_1'' = \frac{E_1}{R_1}, \quad I_2'' = 0.$$

Equations (67) and (68) reduce for this case to the following:

$$I_1 = \frac{E_1}{R_1}\left\{1 + \frac{1}{m_1 - m_2}\left(m_2\epsilon^{\lambda_1 t} - m_1\epsilon^{\lambda_2 t}\right)\right\}. \tag{69}$$

$$I_2 = \frac{E_1}{R_1}\,\frac{m_1 m_2}{m_1 - m_2}\left(\epsilon^{\lambda_1 t} - \epsilon^{\lambda_2 t}\right). \tag{70}$$

The values of m_1, m_2 and λ_1, λ_2 are given by (65). The current in the secondary circuit first commences to rise, reaches a maximum value and then diminishes to zero again. The time in which it reaches its maximum value is given by

$$t = \frac{1}{\lambda_2 - \lambda_1}\log_\epsilon \frac{\lambda_1}{\lambda_2}. \tag{71}$$

As an illustration we may use the same constants as in the previous example,

$$\frac{E_1}{R_1} = 10 \text{ amp.,} \quad m_1 = -0.54, \quad m_2 = 0.74, \quad \lambda_1 = -65, \quad \lambda_2 = -385,$$

$$I_1 = \{10 - 5.78\,\epsilon^{-65\,t} - 4.22\,\epsilon^{-385\,t}\}, \quad I_2 = -3.12\,\{\epsilon^{-65\,t} - \epsilon^{-385\,t}\}.$$

The time in which the secondary will have its maximum value is

$$t = \frac{1}{320} \times 1.78 = 0.0056 \text{ sec.}$$

The table below gives the values of I_1 and I_2 for values of t (0 – 0.015) calculated by formulæ (69) and (70).

t.	I_1.	I_2.
0	0	0
0.001 sec.	1.72	0.80
0.002	2.97	1.29
0.005	5.21	1.80
0.008	6.36	1.71
0.010	6.9	1.56
0.015	7.81	1.16

Oscillation Transformer Having Inductance and Capacity in Series.* — The complete solution of the problem of determining the current and condenser potential in each circuit of the oscillation transformer, when an electric discharge occurs in one circuit, offers considerable mathematical difficulties. It involves the solution of a fourth-degree differential equation of the form

$$C_1C_2(L_1L_2 - M^2)\frac{d^4I}{dt^4} + C_1C_2(L_1R_2 + L_2R_1)\frac{d^3I}{dt^3}$$
$$+ (C_2L_2 + C_1L_1 + C_1C_2R_1R_2)\frac{d^2I}{dt^2} + (C_1R_1 + C_2R_2)\frac{dI}{dt} + I = 0 \qquad (72)$$

where R_1, L_1, C_1 and R_2, L_2, C_2 are the resistances, inductances and capacities of the primary and secondary circuits respectively and M is the mutual inductance between the two circuits. Since, however, in most cases where such circuits are employed, as in the case of a radiotelegraphy, the resistances are small compared with the reactances, we may as an approximation neglect R_1 and R_2 which reduces equation (72) to a second-degree equation. With this limitation, that is, neglecting the resistances, we have for the condenser voltages of the primary and secondary circuits

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* J. A. Fleming, "The Principles of Electric Wave Telegraphy and Telephony," Ch. III.

P. Drude, "Über induktiv Erregung zweier elektrische Schwingungskreise, etc.," *Ann. der Physik*, 1904, Vol. 13, p. 512.

A. Oberbleck, "Über den verlauf der elektrischen Schwingungen bei den Tesla'schen Versuchen," *Wied, Ann. der Physik*, 1895, Vol. 55, p. 623.

Louis Cohen, "Theory of Coupled Circuits having Distributed Inductance and Capacity," *Bulletin Bureau of Standards*, Vol. 5.

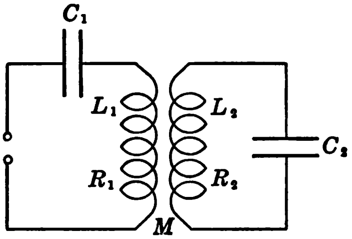

Fig. 61.

$$\left.\begin{aligned} e_1 &= \frac{E_1}{a_2 - a_1}\{a_2 \cos \omega_1 t - a_1 \cos \omega_2 t\}, \\ e_2 &= \frac{E_1 a_1 a_2}{a_2 - a_1}\{\cos \omega_1 t - \cos \omega_2 t\}, \end{aligned}\right\} \tag{73}$$

where
$$a_1 = \frac{1 - \omega_1^2 L_1 C_1}{\omega_1^2 M C_2}, \quad a_2 = \frac{1 - \omega_2^2 L_1 C_1}{\omega_2^2 M C_2}. \tag{74}$$

E_1 is the maximum voltage of condenser C_1, that is, the break-down voltage of the gap, and ω_1, ω_2 are 2π times the frequencies of the circuits. Both circuits oscillate with two periods corresponding to the frequency constants ω_1 and ω_2.

The values of ω_1 and ω_1 are

$$\left.\begin{aligned} \omega_1 &= \sqrt{\frac{C_1 L_1 + C_2 L_2 + \sqrt{(C_1 L_1 - C_2 L_2)^2 + 4 M^2 C_1 C_2}}{2 C_1 C_2 (L_1 L_2 - M^2)}}, \\ \omega_2 &= \sqrt{\frac{C_1 L_1 + C_2 L_2 - \sqrt{(C_1 L_1 - C_2 L_2)^2 + 4 M^2 C_1 C_2}}{2 C_1 C_2 (L_1 L_2 - M^2)}}. \end{aligned}\right\} \tag{75}$$

When the two circuits are syntonized

$$L_1 C_1 = L_2 C_2.$$

Equations (75) reduce to

$$\omega_1 = \sqrt{\frac{1}{LC(1 - K)}}, \quad \omega_2^2 = \sqrt{\frac{1}{LC(1 + K)}}, \tag{76}$$

where $K = \frac{M}{\sqrt{L_1 L_2}}$ and is called the coefficient of coupling.

If we put $CL = \frac{1}{\omega^2}$ where $\frac{2\pi}{\omega}$ is the natural time period of each circuit, we have

$$\omega_1 = \frac{\omega}{\sqrt{1-K}}, \qquad \omega_2 = \frac{\omega}{\sqrt{1+K}}.$$

Substituting the values of a_1 and a_2 from (74) into the second equation of (73) we get

$$e_2 = \frac{E_1 a_1 a_2}{a_2 - a_1} = \frac{E_1 K C_1 \sqrt{L_1 L_2}}{\sqrt{(L_1C_1 - L_2C_2)^2 + 4K^2C_1C_2L_1L_2}}. \tag{77}$$

When the two circuits are syntonized, we get

$$e_2 = \frac{E_1}{2}\sqrt{\frac{C_1}{C_2}}. \tag{78}$$

If $K = 1$ or $M^2 = L_1L_2$ equation (77) becomes

$$e_2 = \frac{E_1C_1\sqrt{L_1L_2}}{L_1C_1 + L_2C_2}$$

and

$$\frac{e_2}{E_1} = \sqrt{\frac{L_1}{L_2}} + \frac{C_2}{C_1}\sqrt{\frac{L_2}{L_1}}. \tag{79}$$

The current in the secondary circuit is also expressed as the sum of two currents of different frequencies.

$$I_2 = \frac{E_1}{2}\sqrt{C_1C_2}\frac{\omega}{\sqrt{1-K}}\sin\frac{\omega}{\sqrt{1-K}}t - \frac{E_1}{2}\sqrt{C_1C_2}\frac{\omega}{\sqrt{1+K}}\sin\frac{\omega}{\sqrt{1+K}}t. \tag{80}$$

The oscillation of the greatest frequency has the greatest amplitude.

As an illustration, we take the following constants:

$L_1 = 2$ mh., $L_2 = 1$ mh., $C_1 = 0.002$ mf., $C_2 = 0.001$ mf., $M = 1$ mh.

We have

$$L_1C_1 = 4 \times 10^{-12}, \quad L_2C_2 = 10^{-12}, \quad 4M^2C_1C_2 = 8 \times 10^{-24},$$
$$C_1C_2(L_1L_2 - M^2) = 2 \times 10^{-18}(2 \times 10^{-6} - 10^{-6}) = 2 \times 10^{-24},$$

hence

$$n_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{5 \times 10^{-12} + \sqrt{9 \times 10^{-24} + 8 \times 10^{-24}}}{4 \times 10^{-24}}}$$
$$= \frac{10^6}{4\pi}\sqrt{5 + \sqrt{17}} = 2.41 \times 10^5,$$

$$n_2 = \frac{\omega_2}{2\pi} = \frac{10^6}{4\pi}\sqrt{5 - \sqrt{17}} = 0.74 \times 10^5.$$

The two frequencies are in the ratio of 3.26 : 1.
If the two circuits are syntonized, that is, if

$$L_2C_2 = L_1C_1 = 4 \times 10^{-12} \quad \text{and} \quad K = 0.5,$$

we have

$$n_1 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{10^{-12}(1-0.5)}} = \frac{10^6}{2\pi\sqrt{0.5}} = 2.25 \times 10^5,$$

$$n_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{10^{-12}(1+0.5)}} = \frac{10^6}{2\pi\sqrt{1.5}} = 1.3 \times 10^5,$$

$$\frac{n_1}{n_2} = 1.73,$$

and the current in the secondary circuit is as follows:

$$I_2 = E_1\left(\frac{2 \times 10^{-9}}{2} \times 2.25 \times 10^5 \sin \omega_1 t - \frac{2 \times 10^{-9}}{2} \times 1.3 \times 10^5 \sin \omega_2 t\right)$$
$$= E_1 \times 10^{-4}\,(2.25 \sin \omega_1 t - 1.3 \sin \omega_2 t).$$

The wave lengths can be obtained from the values of the frequencies by the relation $\lambda = \frac{V}{n}$, where V is the velocity of light;

$$V = 3 \times 10^{10}\,\frac{\text{cm.}}{\text{sec.}}.$$

For the case in the above example when the two circuits are syntonized we have

$$\lambda_1 = \frac{3 \times 10^{10}}{2.25 \times 10^5} = \frac{4}{3} \times 10^5 \text{ cm.} = 1333 \text{ meters.}$$

$$\lambda_2 = \frac{3 \times 10^{10}}{1.3 \times 10^5} = \frac{3}{1.3} \times 10^5 \text{ cm.} = 2307 \text{ meters.}$$

Direct Coupling.—Another form of coupling commonly used in high frequency work for the generation of oscillatory currents is the so-called direct coupling. The arrangement of the circuits is shown in Fig. 62. When an electric discharge occurs in circuit I, the current is partly transmitted to circuit II which in its turn reacts on circuit I and thus both circuits are set in electrical oscillation simultaneously. As in the case of the magnetically-coupled circuits, both circuits oscillate with two different frequencies.

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* G. Seibt, *Physikalische Zeitschrift*, Aug. 1, 1904; or *Écl. Élect.*, Oct. 1, 1904.

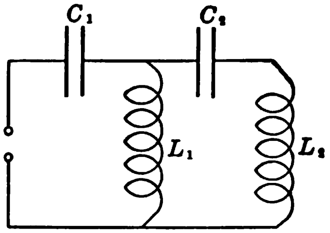

FIG. 62.

If we arrange the constants of the circuits so that both are in syntony we have

$$L_1C_1 = C_2(L_1 + L_2) = CL. \tag{81}$$

Then the frequencies of the oscillations are

$$n_1 = \frac{1}{2\pi}\sqrt{\frac{1}{CL}\left(1 + \frac{L_1}{L_2}\right)\left\{1 + \sqrt{1 - \frac{1}{1 + \frac{L_1}{L_2}}}\right\}}, \tag{82}$$

$$n_2 = \frac{1}{2\pi}\sqrt{\frac{1}{CL}\left(1 + \frac{L_1}{L_2}\right)\left\{1 + \sqrt{1 - \frac{1}{1 + \frac{L_1}{L_2}}}\right\}}. \tag{83}$$

If we write in the above equation $\frac{1}{1 - \rho^2}$ instead of $1 + \frac{L_1}{L_2}$, equations (82) and (83) reduce to

$$n_1 = \frac{1}{2\pi\sqrt{LC}}\,\frac{1}{\sqrt{1 + \rho}}, \tag{84}$$

$$n_2 = \frac{1}{2\pi\sqrt{LC}}\,\frac{1}{\sqrt{1 - \rho}}. \tag{85}$$

If $\frac{L_1}{L_2}$ is small, ρ is also small and the difference in the two frequencies will be also small. As an illustration we may consider the case of antenna circuit coupled directly to the exciting circuit as shown in Fig. 63.

We may use the following constants:

$$L_1 = 0.1 \text{ mh.}, \quad L_2 = 1 \text{ mh.}, \quad C_1 = 0.004 \text{ mf.},$$

$$\frac{L_1}{L_2} = 0.1, \quad 1 - \rho^2 = 0.9, \quad \rho^2 = 0.1 \text{ and } \rho = 0.316.$$

Introducing these values in equations (84) and (85) we get

$$n_1 = \frac{1}{2\pi\sqrt{10^{-4} \times 4 \times 10^{-9}}}\ \frac{1}{\sqrt{1.316}} = 2.2 \times 10^5,$$

$$n_2 = \frac{1}{2\pi\sqrt{10^{-4} \times 4 \times 10^{-9}}}\ \frac{1}{\sqrt{0.684}} = 3.05 \times 10^5.$$

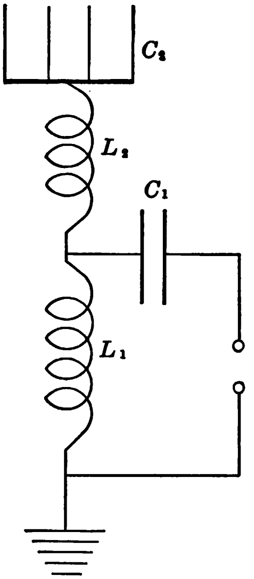

Fig. 63.

The wave lengths corresponding to these frequencies are

$$\lambda_1 = 1364 \text{ meters.}$$
$$\lambda_2 = 984 \text{ meters.}$$

If in the above example we had $L_2 = 2$ mh.,

$$\frac{L_1}{L_2} = 0.05, \quad \frac{1}{1-\rho^2} = 1.05, \quad \rho^2 = 0.05, \quad \rho = 0.224,$$

and

$$n_1 = \frac{1}{2\pi\sqrt{10^{-4} \times 4 \times 10^{-9}}} \cdot \frac{1}{\sqrt{1.224}} = 2.27 \times 10^5,$$

$$n_2 = \frac{1}{2\pi\sqrt{10^{-4} \times 4 \times 10^{-9}}} \cdot \frac{1}{\sqrt{0.776}} = 2.86 \times 10^5.$$

CHAPTER VI

DISTRIBUTED INDUCTANCE AND CAPACITY*

In long-distance transmission problems it is not permissible to assume that the resistance, inductance and capacity are localized at one or more points. The error introduced by such an assumption may be considerable, as will be shown by some examples worked out here. Accurate and reliable results can be obtained only by working out the problem in its most general form, allowing for the uniform distribution of resistance, inductance, capacity and leakage along the entire length of the line. The problem does not offer any great analytical difficulties; it has been discussed by many engineers and physicists and various formulæ have been derived. The numerical work, however, in working out any particular problem by the general formula is somewhat laborious. It is hoped that the various tables and curves worked out here will facilitate considerably the work of numerical computation.

More than one formula are given in many cases so that, by carrying through the computations by two different formulæ, numerical errors may be eliminated.

It is believed that the many examples covering typical cases of transmission lines worked out here will prove to be of value to engineers who have to deal with transmission problems, in giving an approximate idea of the general behavior of a long-distance transmission line in regard to its electrical conditions as to losses, regulations, etc. Furthermore, some examples have also been worked out by other formulæ which have been derived on the assumption of localized inductance and capacity, which will serve to illustrate what errors may be expected in the use of such formulæ.

Notation. — In all the formulæ in this chapter we shall use the following notation:

R = resistance in ohms per unit of length (the mile is a convenient unit of length),

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* A list of references on this subject is given at the end of the chapter.

L = inductance in mh. per unit of length,
C = capacity in mf. per unit of length,
g = leakage conductance in mho. per unit of length,
$\omega = 2\pi \times$ frequency,
n = frequency,
s = distance along the line either from generator end or receiving end,
E_g = voltage at generator end,
I_g = current at generator end,
E_r = voltage at receiving end,
I_r = current at receiving end,
E = voltage at any point on the line,
I = current at any point on the line,
α = attenuation constant,
β = velocity constant.

The General Equation for Current Propagation Along Wires. — The general differential equations for the current and potential distribution along a line of uniformly distributed inductance and capacity are as follows:

$$\left.\begin{aligned} LC\frac{d^2I}{dt^2} + (CR + Lg)\frac{dI}{dt} + RgI &= \frac{d^2I}{ds^2}, \\ LC\frac{d^2E}{dt^2} + (CR + Lg)\frac{dE}{dt} + RgE &= \frac{d^2E}{ds^2}. \end{aligned}\right\} \qquad (1)$$

Since we are to consider only sinusoidal quantities ($I = I_0\epsilon^{j\omega t}$, $E = E_0\epsilon^{j\omega t}$), these equations may be simplified to

$$\left.\begin{aligned} \frac{d^2I}{ds^2} &= V^2I, \\ \frac{d^2E}{ds_2} &= V^2E, \end{aligned}\right\} \qquad (2)$$

where

$$V^2 = -LC\omega^2 + (CR + Lg)\,j\omega + Rg. \qquad (3)$$

The interconnections between E and I are given by the equations:

$$\left.\begin{aligned} (Lj\omega + R)\,I &= \pm\frac{dE}{ds}, \\ (g + Cj\omega) &= \pm\frac{dI}{ds}. \end{aligned}\right\} \qquad (4)$$

The plus sign is used in considering distance from receiving end, that is, in direction of increasing power. The minus sign is used in counting distance from generator end, that is, in the direction of decreasing power.

In equations (2) V is the propagation constant of the line. It is a complex quantity and may be put in the form

$$V = \alpha + j\beta,$$

where

$$\left.\begin{aligned} \alpha &= \sqrt{\tfrac{1}{2}\left\{\sqrt{(R^2+\omega^2L^2)(g^2+C^2\omega^2)}+Rg-\omega^2LC\right\}},\\ \beta &= \sqrt{\tfrac{1}{2}\left\{\sqrt{(R^2+\omega^2L^2)(g^2+C^2\omega^2)}-Rg+\omega^2LC\right\}}.\end{aligned}\right\} \quad (5)$$

α is the attenuation constant giving the rate at which the current and potential waves diminish in intensity as they travel along the line, and is a function of the frequency. The higher the frequency the more rapid the damping. This is very objectionable in telephonic transmission since the higher harmonics of the speech wave are diminished in intensity at a more rapid rate than the lower harmonics, and consequently the wave arriving at the receiving end is distorted.

β is the wave length or velocity constant of the line. If we designate by λ the wave length and by W the velocity of propagation we have

$$W = \frac{2\pi n}{\beta}, \quad \lambda = \frac{2\pi}{\beta}. \quad (6)$$

Example. —

Consider a telephone circuit, metallic return, consisting of No. 14 B. & S. copper wire, interaxial distance 24 ins.

We have

R = 13.3 ohms per mile,
L = 2.21 mh. per mile,
C = 0.0135 mf. per mile.
Assume $g = 0$, leakage negligible.
For $\omega = 1000$, $R^2 + \omega^2L^2 = 181.77$, $C\omega = 13.5 \times 10^{-6}$,

$$\alpha = \sqrt{\frac{13.5}{2} \times 10^{-6}\left(\sqrt{181.77} - 2.21\right)} = 8.72 \times 10^{-3},$$

$$\beta = \sqrt{\frac{13.5}{2} \times 10^{-6}\left(\sqrt{181.77} + 2.21\right)} = 10.29 \times 10^{-3}.$$

The velocity of propagation $W = \frac{\omega}{\beta} = \frac{10^3}{10.3 \times 10^{-3}} = 97{,}100$ miles per second. The wave length $\lambda = \frac{W}{n} = \frac{W \times 2\pi}{\beta} = 609.7$ miles.

If we take

$$\omega = 10{,}000, \qquad R^2 + \omega^2 L^2 = 665.3, \qquad C\omega = 135 \times 10^{-6},$$

$$\alpha = \sqrt{\frac{135}{2} \times 10^{-6}(\sqrt{665.3} - 22.1)} = 15.78 \times 10^{-3},$$

$$\beta = \sqrt{\frac{135}{2} \times 10^{-6}(\sqrt{665.3} + 22.1)} = 56.85 \times 10^{-3},$$

$$W = \frac{\omega}{\beta} = \frac{10^4}{56.85 \times 10^{-3}} = 175{,}900 \text{ miles per second.}$$

$$\lambda = \frac{W}{n} = \frac{W \times 2\pi}{\beta} = \frac{175{,}900 \times 2\pi}{10^4} = 110.5 \text{ miles.}$$

Increasing the frequency ten times has about doubled the value of the attenuation constant, and decreased the wave length.

If we neglect the leakage and make L large by the introduction of loading coils at various points on the line, then we may write equations (5) in the following form:

$$\left.\begin{aligned} \alpha &= \sqrt{\frac{1}{2}C\omega}\sqrt{\omega L\left(1 + \frac{1}{2}\frac{R^2}{\omega^2 L^2} - \frac{1}{4}\frac{R^4}{\omega^4 L^4} + \cdots\right)} \\ &= \frac{1}{2}R\sqrt{\frac{C}{L}}, \text{ approx.} \\ \beta &= \omega\sqrt{CL\left(1 + \frac{R^2}{\omega^2 L^2} + \cdots\right)} = \omega\sqrt{CL}, \text{ approx.} \end{aligned}\right\} \quad (7)$$

The attenuation is practically independent of the frequency.

If the constants of the line are of such magnitude as to make

$$\frac{R}{L\omega} = \frac{g}{C\omega}$$

equations (5) will reduce to

$$\alpha = R\sqrt{\frac{C}{L}} \quad \text{and} \quad \beta = \omega\sqrt{CL\left(1 + \frac{R^2}{\omega^2 L^2}\right)} = \omega\sqrt{CL}, \text{ approx.} \quad (8)$$

In this case the attenuation is also independent of the frequency, but it is to be noticed that in (8) the attenuation constant is twice as large as that in (7).

For telephonic cables when the wires are close together, the inductance is negligibly small, and equations (5) reduce to

$$\left.\begin{aligned}\alpha &= \sqrt{\frac{R}{2}\left(\sqrt{g^2+\omega^2C^2}+g\right)},\\ \beta &= \sqrt{\frac{R}{2}\left(\sqrt{g^2+\omega^2C^2}-g\right)}.\end{aligned}\right\} \qquad (9)$$

When the leakage component is small compared with the capacity component

$$\alpha = \sqrt{\frac{RC\omega}{2}}\left(1+\frac{g}{2\,C\omega}\right). \qquad (10)$$

This is the formula which ordinarily applies to cable circuits employed for telephonic transmission.

We may also express α and β in the following form:

$$\left.\begin{aligned}\alpha &= \frac{\omega \times 3.785}{10^{-6}}\sqrt{\left(1+\frac{1}{4\log_e\frac{D}{r}}\right)\left(\sqrt{1+\frac{R^2}{L^2\omega^2}}-1\right)},\\ \beta &= \frac{\omega \times 3.785}{10^{6}}\sqrt{\left(1+\frac{1}{4\log_e\frac{D}{r}}\right)\left(\sqrt{1+\frac{R^2}{L^2\omega^2}}+1\right)},\end{aligned}\right\} \qquad (11)$$

where D = interaxial distance between the to and fro conductor and r is the radius of the conductor. Since L is a function of $\frac{D}{r}$, we may express α and β, for a given resistance and a given frequency, as a function of $\frac{D}{r}$.

Formulæ (11) were used for the calculation of Tables XVII and XVIII and Figs. 64–67 were plotted from these tables, giving the values of α and β, for the frequencies 25 and 60 cycles, as functions of $\frac{D}{r}$.

TABLE XVII

VALUES OF α AND β PER MILE FOR THE FREQUENCY OF 60 CYCLES

$\frac{D}{r}$.	No. 0000 B. & S. R = 0.26 ohm per mile.		No. 000 B. & S. R = 0.33 ohm per mile.		No. 00 B. & S. R= 0.42 ohm per mile.		No. 1 B. & S. R =0.665 ohm per mile.		No. 3 B. & S. 1.06 ohms per mile.		No. 5 B. & S. R = 1.69 ohms per mile.	
	α	β	α	β	α	β	α	β	α	β	α	β
10	0.829×10^{-3}	2.283×10^{-3}	1.020×10^{-3}	2.357×10^{-3}	1.244×10^{-3}	2.463×10^{-3}	1.757×10^{-3}	2.757×10^{-3}	2.406×10^{-3}	3.211×10^{-3}	3.202×10^{-3}	3.844×10^{-3}
20	0.661×10^{-3}	2.203×10^{-3}	0.819×10^{-3}	2.255×10^{-3}	1.009×10^{-3}	2.330×10^{-3}	1.457×10^{-3}	2.554×10^{-3}	2.031×10^{-3}	2.921×10^{-3}	2.740×10^{-3}	3.453×10^{-3}
30	0.591×10^{-3}	2.172×10^{-3}	0.733×10^{-3}	2.216×10^{-3}	0.909×10^{-3}	2.283×10^{-3}	1.326×10^{-3}	2.476×10^{-3}	1.866×10^{-3}	2.818×10^{-3}	2.537×10^{-3}	3.288×10^{-3}
50	0.521×10^{-3}	2.146×10^{-3}	0.649×10^{-3}	2.180×10^{-3}	0.807×10^{-3}	2.233×10^{-3}	1.190×10^{-3}	2.397×10^{-3}	1.694×10^{-3}	2.684×10^{-3}	2.325×10^{-3}	3.119×10^{-3}
100	0.447×10^{-3}	2.119×10^{-3}	0.560×10^{-3}	2.146×10^{-3}	0.700×10^{-3}	2.186×10^{-3}	1.044×10^{-3}	2.320×10^{-3}	1.508×10^{-3}	2.563×10^{-3}	2.092×10^{-3}	2.944×10^{-3}
200	0.390×10^{-3}	2.102×10^{-3}	0.492×10^{-3}	2.123×10^{-3}	0.617×10^{-3}	2.155×10^{-3}	0.930×10^{-3}	2.265×10^{-3}	1.358×10^{-3}	2.472×10^{-3}	1.905×10^{-3}	2.810×10^{-3}
300	0.365×10^{-3}	2.093×10^{-3}	0.459×10^{-3}	2.112×10^{-3}	0.577×10^{-3}	2.140×10^{-3}	0.874×10^{-3}	2.240×10^{-3}	1.284×10^{-3}	2.429×10^{-3}	1.814×10^{-3}	2.744×10^{-3}
400	0.348×10^{-3}	2.089×10^{-3}	0.439×10^{-3}	2.106×10^{-3}	0.551×10^{-3}	2.132×10^{-3}	0.845×10^{-3}	2.226×10^{-3}	1.236×10^{-3}	2.402×10^{-3}	1.751×10^{-3}	2.711×10^{-3}
600	0.327×10^{-3}	2.082×10^{-3}	0.412×10^{-3}	2.098×10^{-3}	0.519×10^{-3}	2.122×10^{-3}	0.791×10^{-3}	2.205×10^{-3}	1.173×10^{-3}	2.369×10^{-3}	1.672×10^{-3}	2.651×10^{-3}
800	0.314×10^{-3}	2.079×10^{-3}	0.396×10^{-3}	2.092×10^{-3}	0.498×10^{-3}	2.115×10^{-3}	0.761×10^{-3}	2.192×10^{-3}	1.133×10^{-3}	2.347×10^{-3}	1.620×10^{-3}	2.617×10^{-3}
1000	0.304×10^{-3}	2.076×10^{-3}	0.384×10^{-3}	2.089×10^{-3}	0.483×10^{-3}	2.109×10^{-3}	0.739×10^{-3}	2.183×10^{-3}	1.103×10^{-3}	2.332×10^{-3}	1.582×10^{-3}	2.593×10^{-3}

TABLE XVIII

VALUES OF α AND β PER MILE FOR THE FREQUENCY OF 25 CYCLES

$\frac{D}{r}$.	No. 0000 B. & S. $R = 0.26$ ohm per mile.		No. 000 B. & S. $R = 0.33$ ohm per mile.		No. 00 B. & S. $R = 0.42$ ohm per mile.		No. 1 B. & S. $R = 0.665$ ohm per mile.		No. 3 B. & S. $R = 1.06$ ohms per mile.		No. 5 B. & S. $R = 1.69$ ohms per mile.	
	α	β	α	β	α	β	α	β	α	β	α	β
10	0.700×10^{-3}	1.129×10^{-3}	0.827×10^{-3}	1.212×10^{-3}	0.971×10^{-3}	1.314×10^{-3}	1.289×10^{-3}	1.564×10^{-3}	1.688×10^{-3}	1.906×10^{-3}	2.181×10^{-3}	2.353×10^{-3}
20	0.758×10^{-3}	1.049×10^{-3}	0.690×10^{-3}	1.115×10^{-3}	0.818×10^{-3}	1.197×10^{-3}	1.102×10^{-3}	1.407×10^{-3}	1.455×10^{-3}	1.698×10^{-3}	1.891×10^{-3}	2.084×10^{-3}
30	0.528×10^{-3}	1.017×10^{-3}	0.631×10^{-3}	1.076×10^{-3}	0.750×10^{-3}	1.150×10^{-3}	1.019×10^{-3}	1.341×10^{-3}	1.353×10^{-3}	1.609×10^{-3}	1.764×10^{-3}	1.968×10^{-3}
50	0.470×10^{-3}	0.987×10^{-3}	0.568×10^{-3}	1.037×10^{-3}	0.681×10^{-3}	1.102×10^{-3}	0.933×10^{-3}	1.274×10^{-3}	1.246×10^{-3}	1.519×10^{-3}	1.633×10^{-3}	1.849×10^{-3}
100	0.412×10^{-3}	0.956×10^{-3}	0.502×10^{-3}	0.998×10^{-3}	0.605×10^{-3}	1.054×10^{-3}	0.842×10^{-3}	1.206×10^{-3}	1.131×10^{-3}	1.419×10^{-3}	1.489×10^{-3}	1.720×10^{-3}
200	0.367×10^{-3}	0.935×10^{-3}	0.448×10^{-3}	0.970×10^{-3}	0.544×10^{-3}	1.018×10^{-3}	0.755×10^{-3}	1.143×10^{-3}	1.037×10^{-3}	1.348×10^{-3}	1.374×10^{-3}	1.621×10^{-3}
300	0.344×10^{-3}	0.925×10^{-3}	0.422×10^{-3}	0.957×10^{-3}	0.514×10^{-3}	1.001×10^{-3}	0.725×10^{-3}	1.124×10^{-3}	0.990×10^{-3}	1.311×10^{-3}	1.317×10^{-3}	1.572×10^{-3}
400	0.330×10^{-3}	0.919×10^{-3}	0.405×10^{-3}	0.949×10^{-3}	0.495×10^{-3}	0.990×10^{-3}	0.700×10^{-3}	1.107×10^{-3}	0.960×10^{-3}	1.288×10^{-3}	1.279×10^{-3}	1.540×10^{-3}
600	0.312×10^{-3}	0.912×10^{-3}	0.384×10^{-3}	0.939×10^{-3}	0.470×10^{-3}	0.977×10^{-3}	0.668×10^{-3}	1.087×10^{-3}	0.920×10^{-3}	1.257×10^{-3}	1.231×10^{-3}	1.500×10^{-3}
800	0.299×10^{-3}	0.907×10^{-3}	0.370×10^{-3}	0.933×10^{-3}	0.453×10^{-3}	0.969×10^{-3}	0.647×10^{-3}	1.073×10^{-3}	0.895×10^{-3}	1.238×10^{-3}	1.198×10^{-3}	1.473×10^{-3}
1000	0.291×10^{-3}	0.904×10^{-3}	0.359×10^{-3}	0.928×10^{-3}	0.445×10^{-3}	0.964×10^{-3}	0.630×10^{-3}	1.063×10^{-3}	0.876×10^{-3}	1.224×10^{-3}	1.175×10^{-3}	1.454×10^{-3}

TABLE XIX*

VALUES OF α AND β PER MILE FOR TRIANGULAR SPACING OF THREE-PHASE TRANSMISSION LINES AT 25 AND 60 CYCLES

Size of wire, B. & S.	Resistance in ohms per mile	Spacing between wires in inches.	60 Cycles.		25 Cycles.	
			α	β	α	β
No.						
250,000	0.222	72	0.322×10^{-3}	2.11×10^{-3}	0.306×10^{-3}	0.918×10^{-3}
		96	0.306×10^{-3}	2.10×10^{-3}	0.306×10^{-3}	0.912×10^{-3}
		120	0.294×10^{-3}	2.09×10^{-3}	0.283×10^{-3}	0.907×10^{-3}
		144	0.287×10^{-3}	2.09×10^{-3}	0.275×10^{-3}	0.906×10^{-3}
0000	0.263	72	0.374×10^{-3}	2.11×10^{-3}	0.352×10^{-3}	0.932×10^{-3}
		96	0.357×10^{-3}	2.10×10^{-3}	0.337×10^{-3}	0.927×10^{-3}
		120	0.344×10^{-3}	2.10×10^{-3}	0.326×10^{-3}	0.922×10^{-3}
		144	0.334×10^{-3}	2.09×10^{-3}	0.316×10^{-3}	0.915×10^{-3}
000	0.33	72	0.457×10^{-3}	2.12×10^{-3}	0.420×10^{-3}	0.960×10^{-3}
		96	0.437×10^{-3}	2.12×10^{-3}	0.403×10^{-3}	0.953×10^{-3}
		120	0.421×10^{-3}	2.11×10^{-3}	0.390×10^{-3}	0.946×10^{-3}
		144	0.408×10^{-3}	2.11×10^{-3}	0.381×10^{-3}	0.941×10^{-3}
00	0.416	72	0.556×10^{-3}	2.15×10^{-3}	0.497×10^{-3}	0.995×10^{-3}
		96	0.536×10^{-3}	2.14×10^{-3}	0.482×10^{-3}	0.989×10^{-3}
		120	0.517×10^{-3}	2.13×10^{-3}	0.468×10^{-3}	0.980×10^{-3}
		144	0.504×10^{-3}	2.13×10^{-3}	0.456×10^{-3}	0.972×10^{-3}
0	0.525	72	0.678×10^{-3}	2.18×10^{-3}	0.591×10^{-3}	1.05×10^{-3}
		96	0.653×10^{-3}	2.17×10^{-3}	0.569×10^{-3}	1.03×10^{-3}
		120	0.633×10^{-3}	2.16×10^{-3}	0.554×10^{-3}	1.02×10^{-3}
		144	0.615×10^{-3}	2.15×10^{-3}	0.541×10^{-3}	1.01×10^{-3}
1	0.665	72	0.825×10^{-3}	2.23×10^{-3}	0.691×10^{-3}	1.10×10^{-3}
		96	0.791×10^{-3}	2.21×10^{-3}	0.670×10^{-3}	1.09×10^{-3}
		120	0.766×10^{-3}	2.20×10^{-3}	0.655×10^{-3}	1.08×10^{-3}
		144	0.749×10^{-3}	2.19×10^{-3}	0.640×10^{-3}	1.07×10^{-3}
2	0.835	72	0.989×10^{-3}	2.29×10^{-3}	0.800×10^{-3}	1.17×10^{-3}
		96	0.945×10^{-3}	2.27×10^{-3}	0.779×10^{-3}	1.16×10^{-3}
		120	0.920×10^{-3}	2.26×10^{-3}	0.756×10^{-3}	1.14×10^{-3}
		144	0.905×10^{-3}	2.25×10^{-3}	0.744×10^{-3}	1.14×10^{-3}

* This table can also be used for single-phase lines.

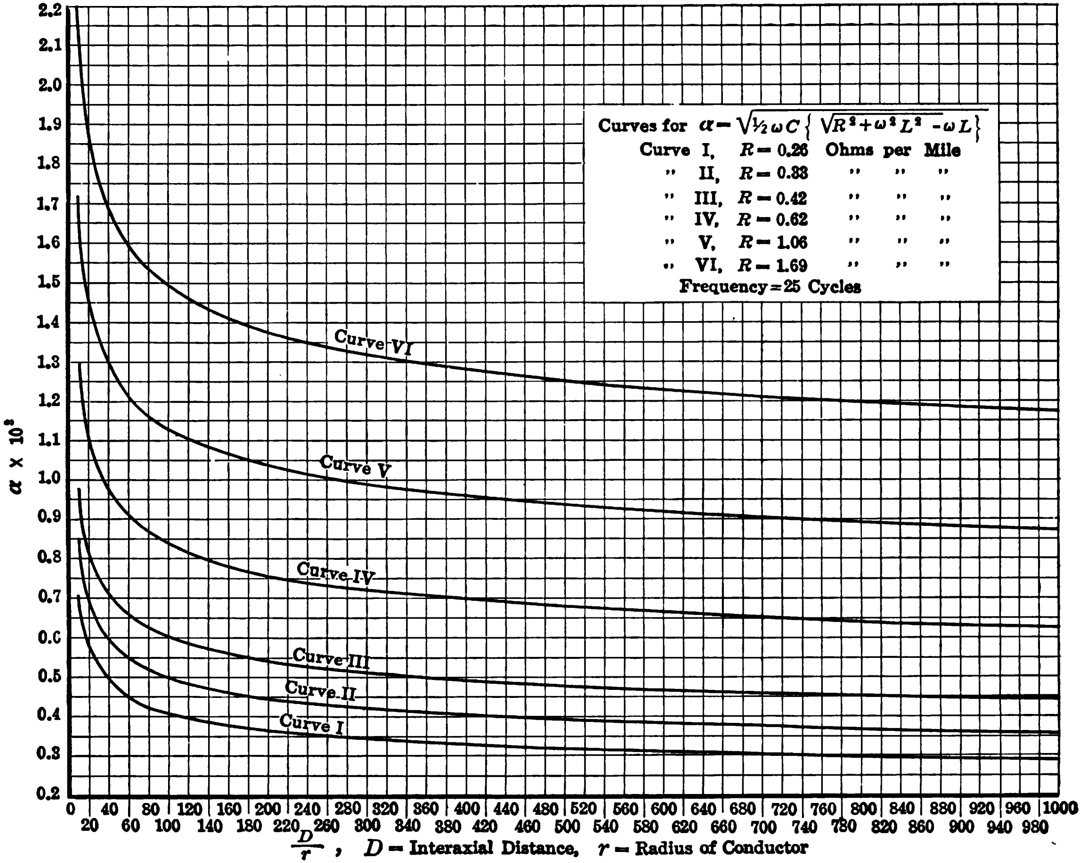

FIG. 64.—The attenuation constant of transmission lines as a function of the interaxial distance and the radius of the conductor.

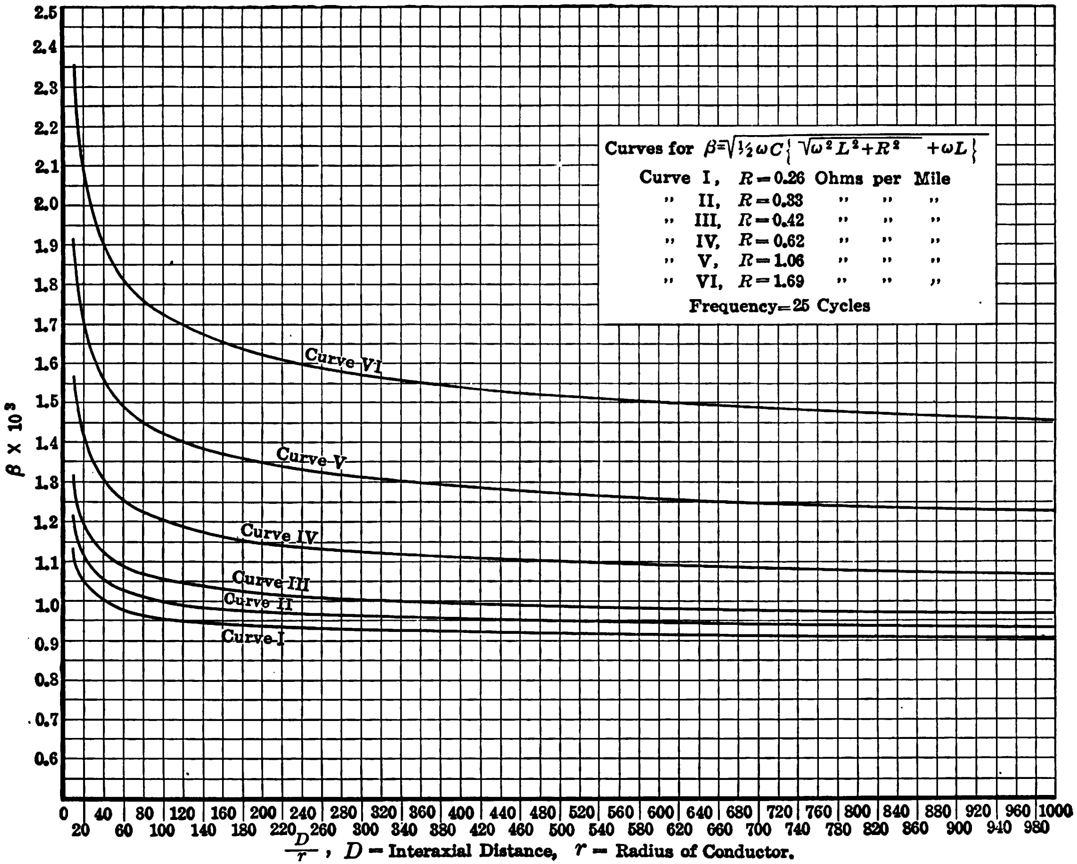

FIG. 65.—The velocity constant of transmission lines as a function of the interaxial distance and the radius of the conductor.

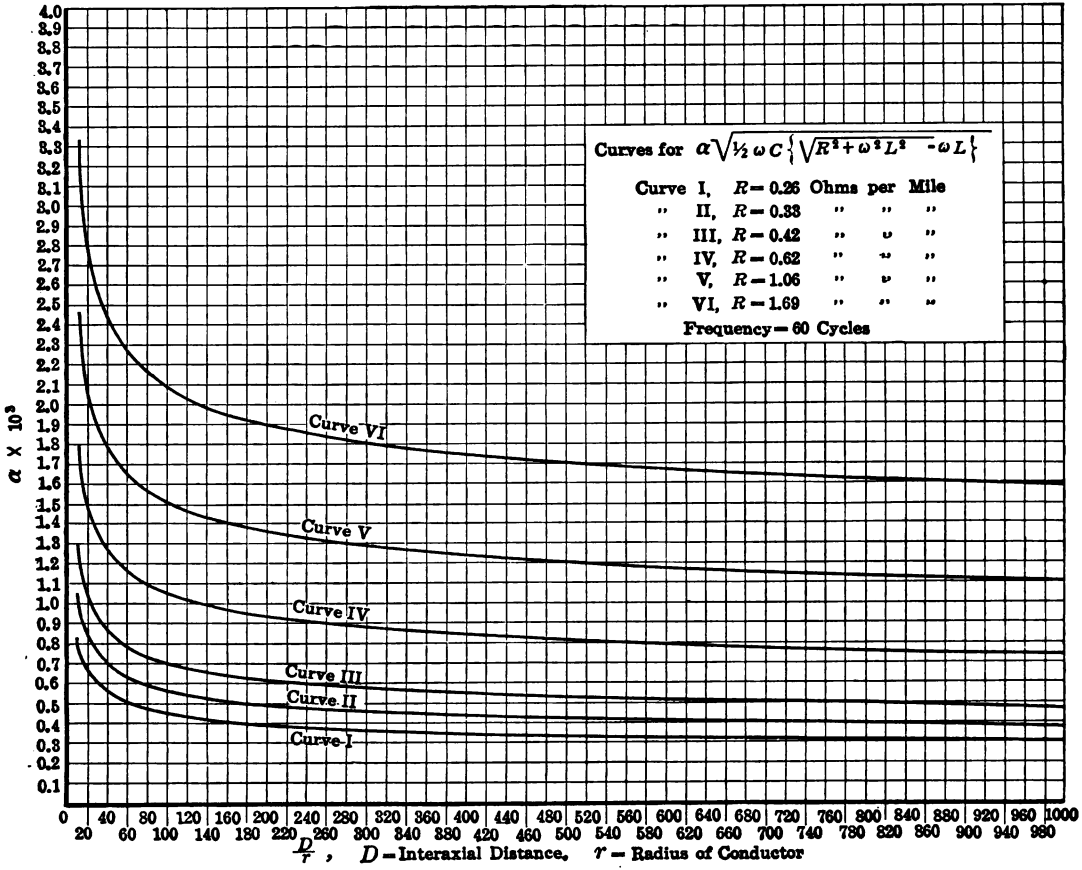

Fig. 66.—The attenuation constant of transmission lines as a function of the interaxial distance and the radius of the conductor.

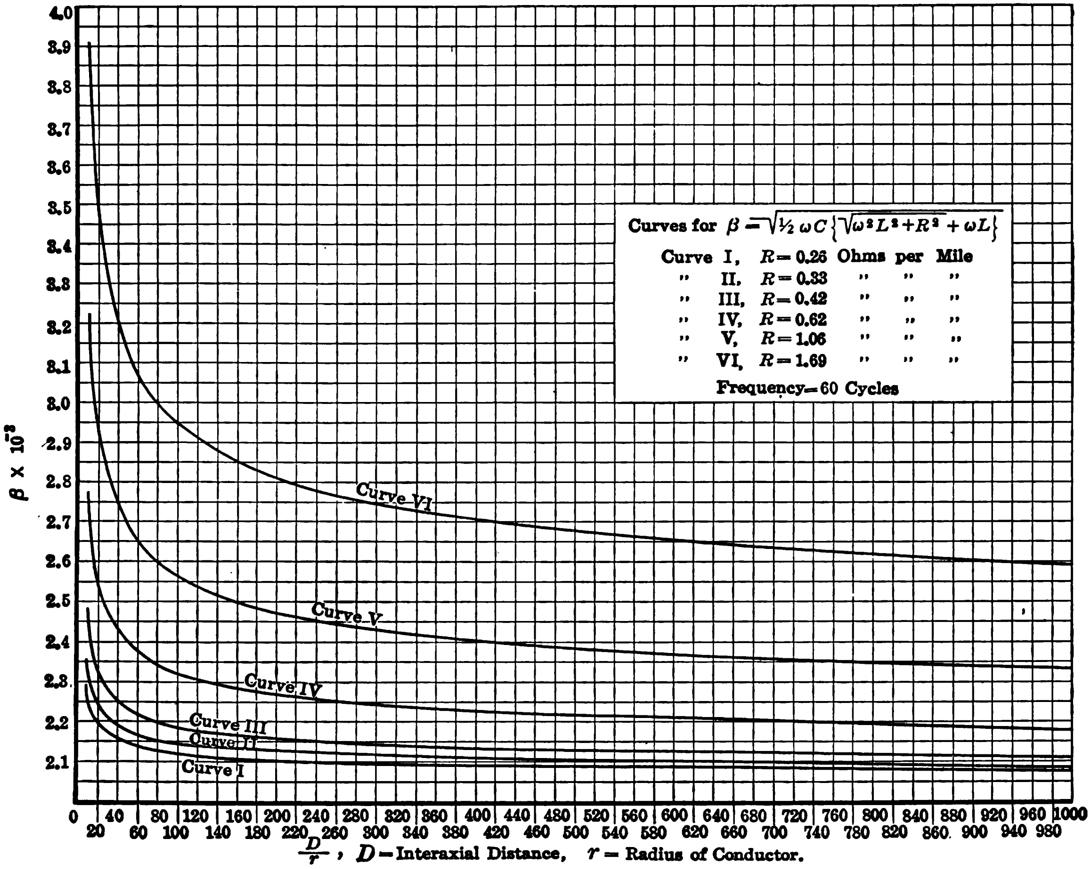

FIG. 67.—The velocity constant of transmission lines as a function of the ratio of the interaxial distance and the radius of the conductor.

The general solution of equations (2) giving the current and voltage at any point on the line is as follows:

$$\left.\begin{aligned} I &= A_1\epsilon^{Vs} + A_2\epsilon^{-Vs}, \\ E &= (a_1 - ja_2)\,(-A_1\epsilon^{Vs} + A_2\epsilon^{-Vs}), \end{aligned}\right\} \qquad (12)$$

or

$$\left.\begin{aligned} I &= A_1\epsilon^{-Vs} + A_2\epsilon^{Vs}, \\ E &= (a_1 - ja_2)\,(-A_1\epsilon^{-Vs} + A_2\epsilon^{Vs}), \end{aligned}\right\} \qquad (13)$$

where

$$a_1 - ja_2 = \frac{V}{Cj\omega + g} = \frac{\alpha + j\beta}{Cj\omega + g},$$

$$\left.\begin{aligned} a_1 &= \frac{\alpha g + \beta C\omega}{g^2 + C^2\omega^2}, \\ a_2 &= \frac{\alpha C\omega - \beta g}{g^2 + C^2\omega^2}. \end{aligned}\right] \qquad (14)$$

The quantity $a_1 - ja_2$ is frequently expressed in the following form:

$$a_1 - ja_2 = \frac{V}{Cj\omega + g} = \frac{\sqrt{(Lj\omega + R)\,(Cj\omega + g)}}{Cj\omega + g} = \frac{\sqrt{Lj\omega + R}}{\sqrt{Cj\omega + g}}, \qquad (15)$$

and is sometimes denoted by Z_0. The quantity Z_0 is called the initial sending-end impedance.

Equations (12) are to be used when the distance is considered from generator end, in the direction of decreasing power, and equations (13) are to be used when the distance is considered from the receiving end, in the direction of increasing power.

The solution of equations (2) may also be expressed in terms of hyperbolic functions.

$$\left.\begin{aligned} I &= A_1 \cosh Vs + A_2 \sinh Vs, \\ E &= -\,(a_1 - ja_2)\,(A_1 \sinh Vs + A_2 \cosh Vs), \end{aligned}\right\} \qquad (16)$$

or

$$\left.\begin{aligned} I &= A_1 \cosh Vs + A_2 \sinh Vs \\ E &= (a_1 - ja_2)\,(A_1 \sinh Vs + A_2 \cosh Vs). \end{aligned}\right\} \qquad (17)$$

Equations (16) are to be used when the distance is considered from the generator end and equations (17) when the distance is considered from the receiving end. The two sets of equations (12), (13) and (16), (17) are equivalent.

The development of the application of hyperbolic functions to the study of electrical problems is mainly due to Prof. A. E.

Kennelly. His researches on this subject were published in a number of very interesting papers.*

The values of the constants A_1, A_2 in equations (12), (13), (16) and (17) are to be determined from the boundary conditions. The evaluation of the constants in most cases and the rearrangement of the final solution so as to put it in the simplest form is a somewhat laborious algebraic process. We shall, therefore, in accordance with the general plan of this work, eliminate all the intermediate steps in the work here, and just give the final solutions for some of the important cases which occur in practice.

INFINITE LINE

The simplest case is that of an infinite line because there is no reflected wave from the receiving end. Let E_g be the generator voltage assumed to be of sine wave form. Then the voltage and current at any point distance s from generator end are

$$\left.\begin{aligned} E &= E_g\epsilon^{-\alpha s}\cos\beta s, \\ I &= \frac{E_g}{\sqrt{a_1^2+a_2^2}}\epsilon^{-\alpha s}\cos(\beta s-\phi) \\ &= E_g\sqrt[4]{\frac{g^2+C^2\omega^2}{R^2+\omega^2L^2}}\epsilon^{-\alpha s}\cos(\beta s-\phi). \end{aligned}\right\} \qquad (18)$$

The ratio $\frac{E}{I} = \sqrt{a_1^2+a_2^2}$ and is the same for every point on the line, that is, the line acts simply as an impedance whose value is $\sqrt{a_1^2+a_2^2}$.

The current leads the voltage by an angle ϕ whose value is

$$\phi = \tan^{-1}\frac{a_2}{a_1} = \tan^{-1}\frac{\alpha C\omega-\beta g}{\alpha g+\beta C\omega} \qquad (19)$$

and which is constant for all points on the line.

If $g = 0$ and $L\omega = 0$, which is approximately the case in cables in which the dielectric losses and the inductance are small, we have

$$\left.\begin{aligned} I &= E_g\sqrt{\frac{C\omega}{R}}\epsilon^{-\alpha s}, \\ \phi &= \tan^{-1}\frac{\alpha}{\beta} = \tan^{-1}1 = 45°. \end{aligned}\right\} \qquad (20)$$

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* References are given at the end of this chapter.

Expressed in hyperbolic functions the formulæ for the current and potential in an infinitely long line are

$$\left.\begin{aligned} I &= \frac{E_g}{a_1 - ja_2}(\sinh Vs - \cosh Vs), \\ E &= E_g(\cosh Vs - \sinh Vs). \end{aligned}\right\} \qquad (21)$$

Since $V = \alpha + j\beta$ is a complex quantity, $\cosh Vs$ and $\sinh Vs$ are complex quantities. If we should separate them into their real and imaginary components, equations (21) would reduce to the same form as equations (18).

LINE OF FINITE LENGTH AND OPEN AT RECEIVING END

Denoting by E_g the impressed voltage and by l the length of the line, the voltage and current at any point on the line distant s from the end of the line are

$$\left.\begin{aligned} E &= \frac{E_g\{\cos\beta s\,(\epsilon^{\alpha s} + \epsilon^{-\alpha s}) + j\sin\beta s\,(\epsilon^{\alpha s} - \epsilon^{-\alpha s})\}}{\cos\beta l\,(\epsilon^{\alpha l} + \epsilon^{-\alpha l}) + j\sin\beta l\,(\epsilon^{\alpha l} - \epsilon^{-\alpha l})}, \\ I &= \frac{E_g}{a_1 - ja_2}\left\{\frac{\cos\beta s\,(\epsilon^{\alpha s} - \epsilon^{-\alpha s}) + j\sin\beta s\,(\epsilon^{\alpha s} + \epsilon^{-\alpha s})}{\cos\beta l\,(\epsilon^{\alpha l} + \epsilon^{-\alpha l}) + j\sin\beta l\,(\epsilon^{\alpha l} - \epsilon^{-\alpha l})}\right\}. \end{aligned}\right\} \qquad (22)$$

The absolute values of E and I at any point on the line are

$$\left.\begin{aligned} E &= E_g\sqrt{\frac{\epsilon^{2\alpha s} + \epsilon^{-2\alpha s} + 2\cos 2\beta s}{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}}, \\ I &= \frac{E_g}{\sqrt{a_1^2 + a_2^2}}\sqrt{\frac{\epsilon^{2\alpha s} + \epsilon^{-2\alpha s} - 2\cos 2\beta s}{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}}. \end{aligned}\right\} \qquad (23)$$

At the end of the line, that is $s = 0$,

$$\left.\begin{aligned} E_r &= \frac{2\,Eg}{\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}}, \\ I_r &= 0. \end{aligned}\right\} \qquad (24)$$

At the generator end of the line, $s = l$,

$$I_g = \frac{E_g}{\sqrt{a_1^2 + a_2^2}}\sqrt{\frac{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}}. \qquad (25)$$

The solution for the distribution of the current and potential along a finite line open at the receiving end, in terms of hyperbolic functions, and considering the distance from the receiving end of the line, is as follows:

$$\left.\begin{aligned} I &= \frac{E_g}{a_1 - ja_2}\,\frac{\sinh Vs}{\cosh Vl}, \\ E &= E_g\frac{\cosh Vs}{\cosh Vl}. \end{aligned}\right\} \qquad (26)$$

We have

$$\left.\begin{aligned} I_r = 0, \quad E_r &= \frac{E_g}{\cosh Vl}, \\ I_g &= \frac{E_g}{a_1 - ja_2}\tanh Vl. \end{aligned}\right\} \qquad (27)$$

Formulæ (22) and (25) are identical, and either can be converted into the other. This can be best shown by a numerical example.

Let us take the same data as given in example worked out on page 202.

$L = 2.21$ mh., $C = 0.0135$ mf., $R = 13.3$ ohms, $g = 0$, $l = 150$ miles.

For the frequency of 1000

$$\alpha = 8.72 \times 10^{-3}, \quad \beta = 10.3 \times 10^{-3}, \quad C\omega = 84.8 \times 10^{-6},$$

$$a_1 = \frac{\beta}{C\omega} = 121, \quad a_2 = \frac{\alpha}{C\omega} = 103.$$

$$\epsilon^{2\alpha l} = 13.69, \quad \epsilon^{-2\alpha l} = 0.073, \quad \cos 2\beta l = -0.998.$$

By formulæ (24) and (25) we get

$$E_r = \frac{2E_g}{\sqrt{13.69 + 0.07 - 1.996}} = 0.58\,E_g.$$

$$I_g = \frac{E_g}{\sqrt{(121)^2 + (103)^2}}\,\frac{\sqrt{13.69 + 0.07 + 1.996}}{\sqrt{13.69 + 0.07 - 1.996}} = 0.007\,E_g.$$

To use the hyperbolic function formulæ we must obtain the values of $\sinh Vl$ and $\cosh Vl$.

We have

$$\sinh Vl = \sinh(\alpha + j\beta)l = \sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l,$$
$$\cosh Vl = \cosh(\alpha + j\beta)l = \cosh \alpha l \cosh \beta l + j \sinh \alpha l \sin \beta l,$$
$$\sinh \alpha l = \sinh 1.308 = 1.714,$$
$$\cosh \alpha l = \cosh 1.308 = 1.985;$$

hence,

$$\sinh Vl = 1.714 \cos \beta l + j\,1.985 \sin \beta l,$$
$$\cosh Vl = 1.985 \cos \beta l + j\,1.714 \sin \beta l.$$

By (27) we get

$$E_r = \frac{E_g}{1.985 \cos \beta l + j\, 1.714 \sin \beta l}.$$

The absolute value is

$$E_r = \frac{E_g}{\sqrt{(1.985)^2 \cos^2 \beta l + (1.714)^2 \sin^2 \beta l}} = 0.58\, E_g$$

$$I_g = \frac{E_g\,(1.714 \cos \beta l + j\, 1.985 \sin \beta l)}{(a_1 - j a_2)(1.985 \cos \beta l + j\, 1.714 \sin \beta l)}$$

$$= E_g \frac{\sqrt{(1.714)^2 \cos^2 \beta l + (1.985)^2 \sin^2 \beta l}}{\sqrt{\{(121)^2 + (103)^2\}\{(1.985)^2 \cos^2 \beta l + (1.714)^2 \sin^2 \beta l\}}}$$

$$= 0.007\, E_g.$$

The agreement in the results obtained by the two methods is exact. For this particular case formulæ (24) and (25) are more suitable for numerical calculations.

SHORT LINES

For short lines, where αl is small, we may put $\epsilon^{\alpha l} = \epsilon^{-\alpha l} = 1$ and equations (23) reduce to

$$\left.\begin{aligned} E_s &= E_g \frac{\cos \beta s}{\cos \beta l}, \\ I_s &= \frac{E_g}{\sqrt{a_1^2 + a_2^2}} \, \frac{\sin \beta s}{\cos \beta l}. \end{aligned}\right\} \tag{28}$$

By equation (14) we have

$$a_1^2 + a_1^2 = \frac{(\alpha g + \beta C\omega)^2 + (\alpha C\omega - \beta g)^2}{(g^2 + C^2\omega^2)^2} = \frac{\alpha^2 + \beta^2}{g^2 + C^2\omega^2}. \tag{29}$$

Introducing the values of α and β, equation (29) reduces to

$$a_1^2 + a_2^2 = \sqrt{\frac{R^2 + L^2\omega^2}{g^2 + C^2\omega^2}}.$$

If we neglect resistance and leakage we have

$$a_1^2 + a_2^2 = \frac{L}{C}. \tag{30}$$

Hence equations (28) become

$$\left.\begin{aligned} E &= E_g \frac{\cos \beta s}{\cos \beta l}, \\ I &= E_g \sqrt{\frac{C}{L}} \, \frac{\sin \beta s}{\cos \beta l}. \end{aligned}\right\} \tag{31}$$

Hence
$$\frac{E}{I} = \sqrt{\frac{L}{C}}.$$

The quantity $\sqrt{\frac{L}{C}}$ is called the natural impedance of the line.

LINE GROUNDED AT RECEIVING END

Assuming here also that we have a sine e.m.f. of amplitude E_g impressed on the line at the generator end, we have the following expression for the current and voltage on the line, considering distance from receiving end.

$$\left.\begin{aligned} I &= \frac{E_g}{a_1 - ja_2}\,\frac{\cos\beta s\,(\epsilon^{\alpha s} + \epsilon^{-\alpha s}) + j\sin\beta s\,(\epsilon^{\alpha s} - \epsilon^{-\alpha s})}{\cos\beta l\,(\epsilon^{\alpha l} - \epsilon^{-\alpha l}) + j\sin\beta l\,(\epsilon^{\alpha l} + \epsilon^{-\alpha l})}, \\ E &= \frac{E_g\{\cos\beta s\,(\epsilon^{\alpha s} - \epsilon^{-\alpha s}) + j\sin\beta s\,(\epsilon^{\alpha s} + \epsilon^{-\alpha s})\}}{\cos\beta l\,(\epsilon^{\alpha l} - \epsilon^{-\alpha l}) + j\sin\beta l\,(\epsilon^{\alpha l} + \epsilon^{-\alpha l})}. \end{aligned}\right\} \quad (32)$$

The absolute values of I and E are

$$\left.\begin{aligned} I &= \frac{E}{\sqrt{a_1^2 + a_2^2}}\sqrt{\frac{\epsilon^{2\alpha s} + \epsilon^{-2\alpha s} + 2\cos 2\beta s}{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}, \\ E &= E_g\sqrt{\frac{\epsilon^{2\alpha s} + \epsilon^{-2\alpha s} - 2\cos 2\beta s}{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}. \end{aligned}\right\} \quad (33)$$

The currents at the generator and receiving ends are

$$\left.\begin{aligned} I_g &= \frac{E_g}{\sqrt{a_1^2 + a_2^2}}\sqrt{\frac{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}, \\ I_r &= \frac{2E_g}{\sqrt{a_1^2 + a_2^2}\,\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}. \end{aligned}\right\} \quad (34)$$

$$\frac{I_r}{I_g} = \frac{2}{\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}. \quad (35)$$

Developing the same problem by the use of hyperbolic functions we obtain the following formulæ:

$$\left.\begin{aligned} I &= \frac{E_g \cosh Vs}{(a_1 - ja_2)\sinh Vl}, \\ E &= \frac{E_g \sinh Vs}{\sinh Vl}. \end{aligned}\right\} \quad (36)$$

The currents at the generator and receiving ends are

$$\left.\begin{aligned} I_g &= \frac{E_g \cosh Vl}{(a_1 - ja_2)\sinh Vl}, \\ I_r &= \frac{E_g}{(a_1 - ja_2)\sinh Vl}. \end{aligned}\right\} \tag{37}$$

$$\frac{I_r}{I_g} = \frac{1}{\cosh Vl} = \operatorname{sech} Vl. \tag{38}$$

As an illustration we may use the same data as given in the problem on page 215.

$$\epsilon^{2\alpha l} = 13.7, \quad \epsilon^{-2\alpha l} = 0.07, \quad \cos 2\beta l = -0.998, \quad a_1 = 121, \quad a_2 = 103.$$

$$\sinh Vl = 1.714 \cos \beta l + j\,1.985 \sin \beta l.$$

$$\cosh Vl = 1.985 \cos \beta l + j\,1.714 \sin \beta l.$$

By (34) we get

$$I_g = \frac{E_g}{\sqrt{(121)^2 + (103)^2}} \frac{\sqrt{13.7 + 0.07 - 1.996}}{\sqrt{13.7 + 0.07 + 1.996}} = 0.0054\,E_g,$$

$$I_r = \frac{2\,E_g}{\sqrt{(121)^2 + (103)^2}\,\sqrt{13.7 + 0.07 + 1.996}} = 0.0037\,E_g.$$

Working out this example by the use of formulæ (37) gives identical results which can be seen by inspection.

LINE SHORT-CIRCUITED AT RECEIVING END

A sine e.m.f. of maximum amplitude E_g is impressed on one end of the line, and the other end is short circuited. Considering distance from receiving end, that is, short-circuited end, we have the following expressions for the current and voltage distribution on the line:

$$\left.\begin{aligned} I &= \frac{E_g\{\cos \beta s\,(\epsilon^{\alpha s} + \epsilon^{-\alpha s}) + j \sin \beta s\,(\epsilon^{\alpha s} - \epsilon^{-\alpha s})\}}{(a_1 - ja_2)\{\cos \beta l\,(\epsilon^{\alpha l} - \epsilon^{-\alpha l}) + j \sin \beta l\,(\epsilon^{\alpha l} + \epsilon^{-\alpha l})\}}, \\ E &= \frac{E_g\{\cos \beta s\,(\epsilon^{\alpha s} - \epsilon^{-\alpha s}) + j \sin \beta s\,(\epsilon^{\alpha s} + \epsilon^{-\alpha s})\}}{\cos \beta l\,(\epsilon^{\alpha l} - \epsilon^{-\alpha l}) + j \sin \beta l\,(\epsilon^{\alpha l} + \epsilon^{-\alpha l})}. \end{aligned}\right\} \tag{39}$$

The absolute values of I and E are

$$\left.\begin{aligned} I &= \frac{E_g \sqrt{\epsilon^{2\alpha s} + \epsilon^{-2\alpha s} + 2 \cos 2\beta s}}{\sqrt{a_1^2 + a_2^2}\,\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2 \cos 2\beta l}}, \\ E &= \frac{E_g \sqrt{\epsilon^{2\alpha s} + \epsilon^{-2\alpha s} - 2 \cos 2\beta s}}{\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2 \cos 2\beta l}}, \end{aligned}\right\} \tag{40}$$

and for $s = l$,

$$I_g = \frac{E_g \sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}}{\sqrt{a_1^2 + a_2^2}\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}. \tag{41}$$

For $s = 0$,

$$\left.\begin{aligned} I_r &= \frac{2E_g}{\sqrt{a_1^2 + a_2^2}\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} - 2\cos 2\beta l}}, \\ E_r &= 0. \end{aligned}\right\} \tag{42}$$

$$\frac{I_r}{I_g} = \frac{2}{\sqrt{\epsilon^{2\alpha l} + \epsilon^{-2\alpha l} + 2\cos 2\beta l}}. \tag{43}$$

Developing the same problem by the use of hyperbolic functions we obtain the following formulæ:

$$\left.\begin{aligned} I &= \frac{E_g \cosh Vs}{(a_1 - ja_2)\sinh Vl}, \\ E &= \frac{E_g \sinh Vs}{\sinh Vl}. \end{aligned}\right\} \tag{44}$$

For $s = l$,

$$I_g = \frac{E_g \coth Vl}{a_1 - ja_2}. \tag{45}$$

For $s = 0$,

$$\left.\begin{aligned} I_r &= \frac{E_g}{(a_1 - ja_2)\sinh Vl}, \\ E_r &= 0. \end{aligned}\right\} \tag{46}$$

$$\frac{I_r}{I_g} = \frac{1}{\cosh Vl} = \text{sech } Vl. \tag{47}$$

Formulæ (44) to (47) are the exact equivalents of formulæ (39) to (43)

POWER TRANSMISSION

In the case of power transmission the load conditions at the receiving end must be taken into consideration. The problem as it frequently presents itself is as follows: Given the voltage and current either at generator or receiving end, to determine the values of voltage and current at the other end of the line.

We shall first consider the case when the voltage and current at the receiving end are known.

Let E_r = voltage at receiving end,
I_r = current at receiving end.

$I_r = I_r' \pm jI_r''$ (a complex quantity), giving the amplitude of the current as well as its phase relation with respect to E_r. The plus sign is to be used when the current is leading and the minus sign when the current is lagging.

Considering the distance s from receiving end, we have the following expressions for the voltage and current at any point on the line:

$$\left.\begin{aligned} I &= \tfrac{1}{2}\{K_1I_r' \mp K_4I_r'' + K_2E_rp - K_3E_rq\} \\ &\quad + \frac{j}{2}\{\pm K_1I_r'' + K_4I_r' + K_2E_rq + K_3E_rp\}, \\ E &= \tfrac{1}{2}\{K_1E_r + K_2(a_1I_r' \pm a_2I_r'') + K_3(a_2I_r' \mp a_1I_r'')\} \\ &\quad + \frac{j}{2}\{K_4E_r + K_2(\pm a_1I_r'' - a_2I_r') + K_3(a_1I_r' \pm a_2I_r'')\}, \end{aligned}\right\} \quad (48)$$

where

$$\left.\begin{aligned} K_1 &= (\epsilon^{\alpha s} + \epsilon^{-\alpha s})\cos\beta s, \\ K_2 &= (\epsilon^{\alpha s} - \epsilon^{-\alpha s})\cos\beta s, \\ K_3 &= (\epsilon^{\alpha s} + \epsilon^{-\alpha s})\sin\beta s, \\ K_4 &= (\epsilon^{\alpha s} - \epsilon^{-\alpha s})\sin\beta s. \end{aligned}\right\} \quad (49)$$

$$p = \frac{a_1}{a_1^2 + a_2^2}, \qquad q = \frac{a_2}{a_1^2 + a_2^2}.$$

The upper sign before I_r'' in equations (48) is to be used when the receiving current is leading the voltage, and the lower sign is to be used when the receiving current is lagging.

To get E_g and I_g, the voltage and current at generator, it is necessary only to put $s = l$ in equations (49).

If we put equations (48) in the form

$$I = I' + jI'',$$
$$E = E' + jE'';$$

the absolute values of I and E are given by

$$\left.\begin{aligned} I &= \sqrt{I'^2 + I''^2}, \\ E &= \sqrt{E'^2 + E''^2}. \end{aligned}\right\} \quad (50)$$

The phase relations between currents and voltages at receiving and generator ends are as follows:

Let ϕ_1 = phase angle between current and voltage at receiving end.

ϕ_2 = phase angle between current at generator end and voltage at receiving end.

ϕ_3 = phase angle between voltages at generator and receiving ends.

$$\left.\begin{aligned} \phi_1 &= \tan^{-1}\frac{I_r''}{I_r'}, \\ \phi_2 &= \tan^{-1}\frac{I_g''}{I_g'}, \\ \phi_3 &= \tan^{-1}\frac{E_g''}{E_g'}. \end{aligned}\right\} \qquad (51)$$

If the values of the tangents of ϕ_1, ϕ_2 and ϕ_3 are positive we have leading phase angles, and if the values are negative we have lagging phase angles.

We have expressions similar to those of (48) to (50) when the current and voltage are given at the generator end. Let E_g denote the voltage at generator end, I_g the current (in intensity and in phase with respect to E_g) at the generator end, that is, $I_g = I_g' \pm jI_g''$. Considering the distance s in this case from the generator end we have

$$\left.\begin{aligned} I &= \tfrac{1}{2}\{K_1I_g' \mp K_4I_g'' - K_2E_gp + K_3E_gq\} \\ &\quad + \frac{j}{2}\{\pm K_1I_g'' + K_4I_g' - K_2E_gq - K_3E_gp\}, \\ E &= \tfrac{1}{2}\{K_1E_g - K_2(a_1I_g' \pm a_2I_g'') - K_3(a_2I_g' \mp a_1I_g'')\} \\ &\quad + \frac{j}{2}\{K_4E_g + K_2(a_2I_g' \mp a_1I_g'') - K_3(a_1I_g' \pm a_2I_g'')\}. \end{aligned}\right\} \qquad (52)$$

The values of K_1, K_2, K_3, K_4 are given by equation (49). The double signs before I_g'' have the same significance as in equation (48). The upper sign is to be used when the generator current is leading the generator voltage by the angle whose tangent is $\frac{I_g''}{I_g'}$ and the lower sign is to be used when the current is lagging.

Equations (48) to (52) give all the formulæ needed in working out most problems in long-distance power transmission.

HYPERBOLIC FORMULÆ

Another set of formulæ corresponding to those of (48) to (52) has been worked out in terms of the hyperbolic functions.

If the electrical conditions are given at the receiving end, considering the distance s from receiving end we have

$$\left.\begin{aligned} I &= I_r \cosh Vs + E_r (p + jq) \sinh Vs, \\ E &= E_r \cosh Vs + I_r (a_1 - ja_2) \sinh Vs. \end{aligned}\right\} \qquad (53)$$

If E_g and I_g are given, that is, the current and voltage at generator end, considering the distance s from generator end we have

$$\left.\begin{aligned} I &= I_g \cosh Vs - E_g (p + jq) \sinh Vs, \\ E &= E_g \cosh Vs - I_g (a_1 - ja_2) \sinh Vs. \end{aligned}\right\} \qquad (54)$$

Formula (53) corresponds to formula (48) and formula (51) corresponds to formula (52), the various terms having the same significance as in the above equations. The terms sinh Vs and cosh Vs are complex quantities. They can be put in the form of two terms, a real and an imaginary, thus,

$$\left.\begin{aligned} \sinh Vs &= \sinh (\alpha + j\beta)\, s = \sinh \alpha s \cos \beta s + j \cosh \alpha s \sin \beta s, \\ \cosh Vs &= \cosh (\alpha + j\beta)\, s = \cosh \alpha s \cos \beta s + j \sinh \alpha s \sin \beta s. \end{aligned}\right\} \qquad (55)$$

The values of sinh αs and cosh αs can be obtained from a table of exponential functions.

A short table, Table XX, is also appended at the end of this chapter giving directly the values of cosh Vs and sinh Vs. The range of values of Vs covered by this table is sufficient for all practical problems in power transmission.

Formulæ (53) and (54) appear simpler and more compact than the corresponding formulæ (48) and (52), but the labor involved in numerical computation by the hyperbolic formulæ is not any less than by the other formulæ, and there is, therefore, little choice in regard to the use of the two different sets of formulæ for numerical problems. It is desirable, however, to be able to check the results by two different methods, so that any possibility of numerical errors may be eliminated.

APPROXIMATE FORMULÆ FOR SHORT LINES

If the electrical conditions are given at the receiving end,

$$\left.\begin{aligned} I &= I_r \left\{ 1 + \frac{s^2 (\alpha^2 - \beta^2)}{2} + j\alpha\beta s^2 \right\} + E_r \{ (p + jq)(\alpha + j\beta) s \}, \\ E &= E_r \left\{ 1 + \frac{s^2 (\alpha^2 - \beta^2)}{2} + j\alpha\beta s^2 \right\} + I_r \{ (a_1 - ja_2)(\alpha + j\beta) s \}. \end{aligned}\right| \qquad (56)$$

If the electrical conditions are given at the generator end,

$$\left.\begin{aligned} I &= I_g\left\{1+\frac{(\alpha^2-\beta^2)\,s^2}{2}+j\alpha\beta s^2\right\}-E_g\{(p+jq)(\alpha+j\beta)s\},\\ E &= E_g\left\{1+\frac{s^2\,(\alpha^2-\beta^2)}{2}+j\alpha\beta s^2\right\}-I_g\{(a_1-ja_2)(\alpha+j\beta)s\}.\end{aligned}\right\} \quad (57)$$

The above two formulæ were given by W. E. Miller.* He states that they can be used with an accuracy of 1 per cent for lines using No. 2 wire up to 120 miles at 60 cycles and 150 miles at 25 cycles. Greater accuracy is obtained if larger wires than No. 2 are used, though the difference is immaterial.

ILLUSTRATIONS

Example. — The following example is taken from Mr. Miller's paper. We shall work it out by the formula (48) and compare the results with those obtained by Mr. Miller by the use of the hyperbolic method.

Three-phase line, 300 miles long using hard-drawn stranded copper wire, No. 000 B. & S. triangularly spaced, with wires 10 ft. apart. Frequency, 60 cycles. At the receiving end we have the following conditions. Line voltage 104 kv. or 60 kv. between wire and neutral; load current 100 amperes at receiving end at 90 per cent power factor lagging. Then

$$E_r = 60\,\text{kv.},$$

$$I_r = I_r' - jI_r'' = 90 - j\,43.5 \text{ amperes},$$

$$R = 0.33 \text{ ohm per mile}, \qquad L = 2.13 \text{ mh. per mile},$$

$$C = 0.014 \text{ mf. per mile}, \qquad g = 0 \text{ (neglected)},$$

$$\alpha = 0.421 \times 10^{-3}, \qquad \beta = 2.11 \times 10^{-3},$$

$$\alpha l = 0.126, \qquad \beta l = 0.633 = 36^\circ\,16'\,5'',$$

$$\epsilon^{\alpha l} + \epsilon^{-\alpha l} = 2.016, \qquad \cos\beta l = 0.8062,$$

$$\epsilon^{\alpha l} - \epsilon^{-\alpha l} = 0.253, \qquad \sin\beta l = 0.5915,$$

$$K_1 = 1.625, \quad K_2 = 0.204, \quad K_3 = 1.193, \quad K_4 = 0.149,$$

$$a_1 = 392, \quad a_2 = 78, \quad p = 2.44 \times 10^{-3}, \quad q = 0.485 \times 10^{-3}.$$

Introducing these values in equation (48) we get for current and voltage at generator end

$$I_g = 73.94 + j\,61.65 \text{ amp.},$$

$$E_g = 66.4 + j\,21.04\,\text{kv.}$$

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* W. E. Miller, Formulæ, Constants and Hyperbolic Functions for Transmission Line Problems, *General Electric Review Supplement*, May, 1910.

The absolute values of current and voltage are

$$I_g = \sqrt{(73.94)^2 + (61.65)^2} = 96.3 \text{ amp.}$$
$$E_g = \sqrt{(66.4)^2 + (21.04)^2} = 69.6 \text{ kv.}$$

The generator current leads the voltage at the receiving end by an angle

$$\tan^{-1}\frac{61.65}{73.94} = \tan^{-1} 0.834 = 39^\circ\, 48'.$$

The generator voltage leads the voltage at the receiving end by an angle

$$\tan^{-1}\frac{21.04}{63.36} = \tan^{-1} 0.317 = 17^\circ\, 36'.$$

The current at the generator end leads the generator voltage by the angle

$$(39^\circ\, 48' - 17^\circ\, 36') = 22^\circ\, 12'.$$

The power factor at the generator end is

$$\cos\,(22^\circ\, 12') = 0.926.$$

Transmission efficiency,

$$\eta = \frac{60 \times 100 \times 0.9}{69.6 \times 96.3 \times 0.926} = 0.88.$$

To get the value of the charging current put $I_r = 0$, no load current.

Then $\quad I_g = -\,2.43 + j\,90.3 = 90.3$ amperes per wire.

The values obtained by Mr. Miller for this example using the hyperbolic functions are

$$I_g = 74.6 + j\,61.9 \text{ amp.,} \qquad E_g = 66.2 + j\,21.0 \text{ kv.}$$

The values differ very slightly from the values obtained by formula (48) and even this slight difference may be accounted for by the fact that in using the hyperbolic tables, interpolations are required, and apparently Mr. Miller only used first differences.

It is interesting to obtain the generator current and voltage and the different phase angles for different load currents. Since we have given in detail the numerical work for the case of 100 amperes load current, it will be sufficient to simply tabulate the results for other load currents.

I_r.	25	50	100	150
$I_g'+jI_g''$	$16.7+j\,83.1$	$35.9+j\,74.9$	$73.9+j\,61.65$	$112.1+j\,47.3$
$E_g'+jE_g''$	$53.2+j\,8.6$	$57.9+j\,12.6$	$66.4+j\,21.04$	$75.2+j\,29.3$
$\sqrt{I_g'^2+I_g''^2}$	84.8 amp.	83 amp.	96.3 amp.	121.7 amp.
$\sqrt{E_g'^2+E_g''^2}$	53.8 kv.	59.2 kv.	69.6 kv.	80.7 kv.
Phase angle, I_gE_r	78° 40′ leading	64° 18′ leading	39° 48′ leading	22° 54 leading
Phase angle, E_gE_r	9° 12′ leading	12° 18′ leading	17° 36′ leading	21° 16′ leading
Phase angle, I_gE_g	69° 28′ leading	52° leading	22° 12′ leading	1° 38′ leading
η = efficiency of transmission	84%	89.2%	88%	82.5%

As an illustration of the application of formulæ (52) and (54) we shall take the following example:

A three-phase line made up of No. 0000 B. & S. copper-strand wire triangularly spaced with 10 feet between wires; frequency, 60 cycles; length of line 200 miles.

Voltage at generator between wires, 104 kv., between wire and neutral, 60 kv.

$$I_g = 120 \text{ amperes at } .90 \text{ per cent power factor leading.}$$

$$I_g = 108 + j\,52.3,$$

$$E_g = 60.$$

To determine the voltage and current at receiving end and the various phase angles.

$$R = 0.26 \text{ ohm per mile.}$$

$$L = 2.09 \text{ mh. per mile.}$$

$$C = 0.0143 \text{ mf. per mile.}$$

$$\alpha = 0.344, \quad \beta = 2.10 \times 10^{-3}.$$

$$\beta l = 0.420 = 24° 3' 50'',$$

$$\cos\beta l = 0.913, \quad \sin\beta l = 0.408,$$

$$\epsilon^{\alpha l} + \epsilon^{-\alpha l} = 2.005, \quad \epsilon^{\alpha l} - \epsilon^{-\alpha l} = 0.1376,$$

$$K_1 = 1.831, \quad K_2 = 0.126, \quad K_3 = 0.818, \quad K_4 = 0.0561,$$

$$a_1 = 396, \quad a_2 = 62.5,$$

$$p = 2.47 \times 10^{-3}, \quad q = 0.39 \times 10^{-3}.$$

Introducing the above values in equations (52) we get

$$I_r = 97.64 - j\,11.18 \text{ amp.},$$

$$E_r = 57.74 - j\,18.02 \text{ kv.}$$

Using formulæ (54) for this example, we find from Table XX by interpolation

$$\cosh Vl = \cosh (0.0688 + j\,0.42) = 0.915 + j\,0.028,$$
$$\sinh Vl = \sinh (0.0688 + j\,0.42) = 0.063 + j\,0.409,$$

and using the values of p, q, a_1 and a_2 as obtained above we get

$$I_r = 97.6 - j\,11.19.$$
$$E_r = 57.7 - j\,17.98.$$

The agreement in the values of I_r, E_r as calculated by the two different formulæ is very close indeed.

The absolute values of the current and voltage are

$$I_r = \sqrt{(97.6)^2 + (11.2)^2} = 98.2 \text{ amp.},$$
$$E_r = \sqrt{(57.7)^2 + (18)^2} = 60.4 \text{ kv.}$$

The current at receiving end lags behind voltage at generator end by angle $\tan^{-1} \frac{11.2}{97.6} = \tan^{-1} 0.101 = 5° \, 46'$.

The voltage at receiving end lags behind voltage at generator end by angle $\tan^{-1} \frac{18.02}{57.7} = \tan^{-1} 0.312 \doteq 17° \, 20'$.

Therefore, the current at receiving end leads the voltage at receiving end by the angle

$$17° \, 20' - 5° \, 46' = 11° \, 34'.$$

$$\text{Transmission efficiency} = \frac{98.2 \times 60.4 \times 0.98}{120 \times 60 \times 0.90} = 0.89 \text{ per cent.}$$

GENERATOR VOLTAGE AND IMPEDANCE AT RECEIVING END KNOWN

The problem in long distance transmission may also present itself in the following form: Given the potential at the generator end and the impedance at the receiving end, to determine the current at the generator end and the potential and current at the receiving end.

Let E_g denote the voltage at the generator end,
Z denote impedance at the receiving end.

Distance is considered from receiving end.

The values of I_g, E_r, and I_r are given by the following formulæ:

$$\left.\begin{aligned} E_r &= \frac{E_g\,(q-jh)\,(c-1+jd)}{q^2+h^2}, \\ I_r &= \frac{E_g\,(q-jh)\,(c+1+jd)\,(a_1+ja_2)}{(a_1{}^2+a_2{}^2)\,(q^2+h^2)}, \\ I_g &= \frac{E_g\,(K_1+jK_2)\,(a_1+ja_2)\,(q-jh)}{(a_1{}^2+a_2{}^2)\,(q^2+h^2)}, \end{aligned}\right\} \qquad (58)$$

where

$$a_1 = \frac{\beta}{C\omega}, \quad a_2 = \frac{\alpha}{C\omega},$$

$$c+jd = \frac{a_1-ja_2+Z}{a_1-ja_2-Z}. \qquad (59)$$

$$\begin{aligned} q &= c\epsilon^{\alpha l}\cos\beta l - d\epsilon^{\alpha l}\sin\beta l - \epsilon^{-\alpha l}\cos\beta l, \\ h &= c\epsilon^{\alpha l}\sin\beta l + d\epsilon^{\alpha l}\cos\beta l + \epsilon^{-\alpha l}\sin\beta l, \\ K_1 &= c\epsilon^{\alpha l}\cos\beta l - d\epsilon^{\alpha l}\sin\beta l + \epsilon^{-\alpha l}\cos\beta l, \\ K_2 &= c\epsilon^{\alpha l}\sin\beta l + d\epsilon^{\alpha l}\cos\beta l - \epsilon^{-\alpha l}\sin\beta l. \end{aligned}$$

In terms of hyperbolic functions we obtain the following set of formulæ for this problem:

$$\left.\begin{aligned} I &= \frac{E_g\left\{\cosh Vs + \frac{Z}{Z_0}\sinh Vs\right\}}{Z_0\sinh Vl + Z\cosh Vl}, \\ E &= \frac{E_g\left\{Z_0\sinh Vs + Z\cosh Vs\right\}}{Z_0\sinh Vl + Z\cosh Vl}, \end{aligned}\right\} \qquad (60)$$

where Z_0 has been put for brevity in place of $a_1 - ja_2$.

For I and E at the receiving end, we put $s = 0$ in formulæ (60), and we get

$$\left.\begin{aligned} I_r &= \frac{E_g}{Z_0\sinh Vl + Z\cosh Vl}, \\ E_r &= \frac{E_g Z}{Z_0\sinh Vl + Z\cosh Vl}. \end{aligned}\right\} \qquad (61)$$

At the generator end, $s = l$, we have

$$I_g = \frac{E_g\left\{\cosh Vl + \frac{Z}{Z_0}\sinh Vl\right\}}{Z_0\sinh Vl + Z\cosh Vl}. \qquad (62)$$

Example. —

Let $Z = 540 + j\,261, \qquad \sqrt{(540)^2 + (261)^2} = 600,$ and let us take for the line constants the same as those given in the example on page 223.

$$R = 0.33 \text{ ohm per mile,}$$
$$L = 2.13 \text{ mh. per mile,}$$
$$C = 0.014 \text{ mf. per mile,}$$
$$g = 0, \quad l = 300 \text{ miles,} \quad \omega = 2\pi \times 60, \quad E_g = 60 \text{ kv.,}$$
$$\alpha l = 0.126, \quad \beta l = 0.633,$$
$$\epsilon^{\alpha l} = 1.134, \; \epsilon^{-\alpha l} = 0.882, \; \cos\beta l = 0.806, \; \sin\beta l = 0.5915,$$
$$a_1 = 392, \qquad a_2 = 78,$$
$$c + jd = \frac{392 - j\,78 + 540 + j\,261}{392 - j\,78 - 540 - j\,261} = -1.46 + j\,2.11,$$
$$q = -3.46, \quad h = 1.47, \quad K_1 = -2.04, \quad K_2 = 0.43,$$
$$q^2 + h^2 = 14.13, \quad (a_1^2 + a_2^2)(q^2 + h^2) = 2{,}257{,}240.$$

Introducing these values in formulæ (58) we get

$$E_r = \frac{E_g(-3.46 - j\,1.47)(-2.46 + j\,2.11)}{14.13} = E_g(0.82 - j\,0.26)$$
$$= 49.2 - j\,15.6 \text{ kv.}$$

The absolute value of E_r is $\sqrt{(49.2)^2 + (15.6)^2} = 51.6$ kv.
The voltage at receiving end lags behind the generator voltage by the angle

$$\tan^{-1}\frac{15.6}{49.2} = \tan^{-1} 0.317 = 17^\circ\, 36'.$$

$$I_r = \frac{E_g(-3.46 - j\,1.47)(-0.46 + j\,2.11)(392 + j\,78)}{2{,}257{,}240}$$
$$= \frac{E_g(1.043 - j\,0.987)}{1000} = 62.6 - j\,59.5 \text{ amp.,}$$

absolute value = $\sqrt{(62.6)^2 + (59.5)^2} = 86.4$ amp.

The current at receiving end lags behind the generator voltage by the angle

$$\tan^{-1}\frac{59.5}{62.6} = \tan^{-1} 0.934 = 43^\circ\, 3'.$$

$$I_g = \frac{E_g(-2.04 + j\,0.43)(392 + j\,78)(-3.46 - j\,1.47)}{2{,}257{,}240}$$
$$= \frac{E_g(1.28 + j\,0.528)}{1000} = 76.8 + j\,31.7 \text{ amp.,}$$

absolute value = $\sqrt{(76.8)^2 + (31.7)^2} = 83.1$ amp.

The current at the generator end leads the generator voltage by the angle

$$\tan^{-1}\frac{31.68}{76.8} = \tan^{-1} 0.413 = 22^\circ\, 27'.$$

The current at the receiving end lags behind the voltage at the receiving end by the angle

$$43^\circ\, 3' - 17^\circ\, 36' = 24^\circ\, 27'.$$

At the generator end the power factor is cos (22° 27′) = 0.924 and at the receiving end the power factor is cos (24° 27′) = 0.910.

Efficiency of transmission,

$$\eta = \frac{51.6 \times 86.4 \times 0.910}{60 \times 83.1 \times 0.924} = 88 \text{ per cent.}$$

Working out the same problem by the hyperbolic formulæ we get

$$\begin{aligned} \cosh Vl &= \cosh(\alpha l + j\beta l) = \cosh(0.126 + j\,0.633) \\ \sinh Vl &= \sinh(\alpha l + j\beta l) = \sinh(0.126 + j\,0.633). \end{aligned}$$

From Table XX we find

$$\begin{aligned} \cosh(0.12 + j\,0.64) &= 0.808 + j\,0.072, \\ \cosh(0.12 + j\,0.62) &= 0.820 + j\,0.070, \\ \cosh(0.14 + j\,0.64) &= 0.810 + j\,0.083, \\ \cosh(0.14 + j\,0.62) &= 0.822 + j\,0.081. \end{aligned}$$

Therefore,

$$\begin{aligned} \cosh(0.12 + j\,0.633) &= 0.812 + j\,0.072, \\ \cosh(0.14 + j\,0.633) &= 0.814 + j\,0.082, \end{aligned}$$

and

$$\cosh(0.126 + j\,0.633) = 0.812 + j\,0.075.$$

In a similar way we obtain from the same table by interpolation

$$\sinh(0.120 + j\,0.633) = 0.102 + j\,0.597,$$

$$Z_0 = a_1 - ja_2 = 392 - j\,78, \quad Z = 540 + j\,261, \quad E_g = 60 \text{ kv.}$$

Introducing the above values in equations (61) and (62) we get

$$\begin{aligned} I_r &= \frac{60{,}000}{(392 - j\,78)(0.102 + j\,0.597) + (540 + j\,261)(0.812 + j\,0.075)} \\ &= \frac{60{,}000}{505.4 + j\,478.5} \\ &= \frac{60{,}000\,(505.4 - j\,478.5)}{484{,}391} = 62.6 - j\,59.3 \text{ amp.} \end{aligned}$$

$$I_g = \frac{E_g\left\{0.812 + j\,0.075 + \dfrac{540 + j\,261}{392 - j\,78}(0.102 + j\,0.597)\right\}}{(392 - j\,78)(0.102 + j\,0.597) + (540 + j\,261)(0.812 + j\,0.075)}$$

$$= \frac{E_g\{0.812 + j\,0.075 + (1.198 + j\,0.904)(0.102 + j\,0.597)\}}{(392 - j\,78)(0.102 + j\,0.597) + (540 + j\,261)(0.812 + j\,0.075)}$$

$$= \frac{60{,}000\,(0.394 + j\,0.882)}{505.4 + j\,478.5} = 76.7 + j\,31.8 \text{ amp.}$$

$$E_r = \frac{E_g\,(540 + j\,261)}{505.4 + j\,478.5} = 49.2 - j\,15.6 \text{ kv.}$$

The results obtained by the two different sets of formulæ are in exact agreement.

TRANSIENT PHENOMENA IN CIRCUITS HAVING DISTRIBUTED INDUCTANCE AND CAPACITY

In Chapter V we gave a brief discussion of transient electrical phenomena. We have shown that if the circuit conditions are disturbed in any way the current and potential do not pass over instantly from one permanent value to another; it always requires a certain time interval before the current and potential reach their steady values again. During the transition period the current and potential may be considered as consisting of two components, one of which corresponds to the permanent value, and the other the transient term which disappears more or less rapidly.

In the formulæ given in Chapter V we have considered only circuits of localized inductance and capacity, so that the space element does not enter into the formulæ. We were concerned only with the time element, the time rate at which the transient term disappears.

In problems, however, connected with long-distance transmission lines, the distributed character of the inductance, capacity and resistance must be taken into consideration. If, for instance, an e.m.f. is impressed on a long line, every element of the line is not charged to the same potential at the same instant. It requires a certain length of time before the entire line is completely charged, and during that interval the charge and potential differ for different points on the line. In the discussion, therefore, of transient phenomena on lines we must deal with two variables, distance and time.

It is to be regretted that the formulæ for transient phenomena on transmission lines cannot be presented in the same manner as those given in other chapters. The formulae are generally very long and involved, and if the bare formulæ are given leaving out the derivation and the complete discussion of the physical significance of the various factors, their usefulnesss is greatly reduced. The formulæ for transient phenomena occurring on a transmission line have their greatest value in enabling us to get a proper appreciation of the phenomena occurring on the line irrespective of magnitude.

It was thought advisable, therefore, not to attempt to cover thoroughly this branch of the subject, but to give only a few formulæ for the cases of most common occurence. Those interested in the subject can refer to Professor Steinmetz's book "Transient Electric Phenomena and Oscillations" where the whole subject is discussed in a masterly manner.

TRANSIENT PHENOMENA IN CIRCUITS HAVING DISTRIBUTED RESISTANCE AND CAPACITY

Assume the inductance of the line negligible as in the case of transatlantic cables.

Suppose the cable is open at receiving end, and a constant continuous e.m.f. E is suddenly applied at the sending end. A charging current commences to flow in the cable and propagates itself from point to point on the cable until the whole cable is charged to the impressed potential E. During the charging period the current and potential vary at different points on the cable, so that the current and potential are functions of the time and the distance.

Let r = resistance per unit length,
C = capacity per unit length,
l = length of cable,
E = impressed e.m.f,
s = distance from sending end to any point on the cable.

The potential at any point on the cable distant s from sending end may be put in the form,

$$e = E + e_v, \tag{63}$$

where

$$e_v = -\frac{4E}{\pi}\left\{\epsilon^{-\frac{\pi^2}{l^2}\frac{t}{4rC}}\sin\left(\frac{\pi}{2}\frac{s}{l}\right) + \frac{1}{3}\epsilon^{-\frac{\pi^2}{l^2}\frac{9t}{4rC}}\sin\left(\frac{3\pi}{2}\frac{s}{l}\right)\right.$$
$$\left. + \frac{1}{5}\epsilon^{-\frac{\pi^2}{l^2}\frac{25t}{4rC}}\sin\left(\frac{5\pi}{2}\frac{s}{l}\right) + \cdots\right\}. \qquad (64)$$

The current

$$i_v = \frac{2E}{lr}\left\{\epsilon^{-\frac{\pi^2}{l^2}\frac{t}{4rC}}\cos\left(\frac{\pi}{2}\frac{s}{l}\right) + \epsilon^{-\frac{\pi^2}{l^2}\frac{9t}{4rC}}\cos\left(\frac{3\pi}{2}\frac{s}{l}\right)\right.$$
$$\left. + \epsilon^{-\frac{\pi^2}{l^2}\frac{25t}{4rC}}\cos\left(\frac{5\pi}{2}\frac{s}{l}\right) + \cdots\right\}. \qquad (65)$$

If we put $lr = r_1$ total resistance,
$lC = C_1$ total capacity,

then, neglecting upper harmonics, formulæ (64) and (65) become approximately,

$$\left.\begin{aligned} e_v &= -\frac{4E}{\pi}\epsilon^{-\frac{\pi^2}{4r_1C_1}t}\sin\left(\frac{\pi}{2}\frac{s}{l}\right), \\ i_v &= \frac{2E}{r_1}\epsilon^{-\frac{\pi^2}{4r_1C_1}t}\cos\left(\frac{\pi}{2}\frac{s}{l}\right). \end{aligned}\right\} \qquad (66)$$

At the end of the line ($s = l$), the potential is x per cent less than the maximum at the time T_x given by

$$\left.\begin{aligned} \frac{x}{100} &= \frac{4}{\pi}\epsilon^{-\frac{\pi^2}{4r_1C_1}T_x}, \\ T_x &= \frac{4r_1C_1}{\pi^2}\log_e\frac{400}{\pi x}. \end{aligned}\right\} \qquad (67)$$

As an illustration we may take the cable between Ireland and Newfoundland which is 2640 km. long. The total resistance $r_1 = 6000$ ohms, $C_1 = 40$ mf.

The problem is to determine the time required before e reaches the value $0.95\,E$.

We have $e_v = 0.05\,E, \quad x = 5,$

$$\because \quad T_x = \frac{4r_1C_1}{\pi^2}\log_e\frac{400}{5\pi} = \frac{4 \times 6000 \times 40 \times 10^{-6}}{\pi^2}\log_e\frac{80}{\pi}$$
$$= \frac{96 \times 10^{-2}}{\pi^2} \times 3.27 = 0.31 \text{ second.}$$

The time required for the potential to attain 99 per cent of its normal value is

$$T_x = \frac{4 \times 6000 \times 40 \times 10^{-6}}{\pi^2} \log_\epsilon \frac{400}{\pi} = 0.499 \text{ second.}$$

As another illustration let us determine the value of the potential and current for points 100 km., 1000 km. and 2000 km. distant at the time $t = 0.1$ sec. Neglecting the upper harmonics and using formula (66) we have

$$\frac{\pi^2}{4 r_1 C_1} = \frac{9.87}{4 \times 6000 \times 40 \times 10^{-6}} = 10.28.$$

$$\epsilon^{-\frac{\pi^2}{4 r_1 C_1} t} = \epsilon^{-1.028} = 0.358.$$

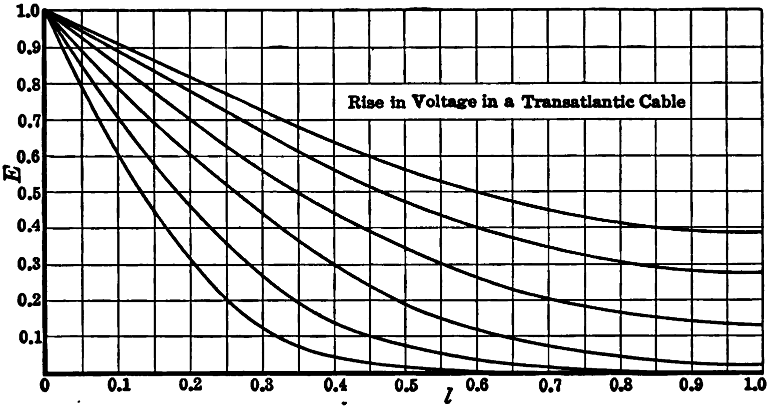

FIG. 68.—Rise in voltage in a transatlantic cable.

For

$$s = 100 \text{ km.}, \quad \sin\left(\frac{\pi}{2}\frac{s}{l}\right) = \sin(0.06) = 0.06, \quad \cos(0.06) = 0.998,$$

$$s = 1000 \text{ km.}, \quad \sin\left(\frac{\pi}{2}\frac{s}{l}\right) = \sin(0.60) = 0.56, \quad \cos(0.60) = 0.825,$$

$$s = 2000 \text{ km.}, \quad \sin\left(\frac{\pi}{2}\frac{s}{l}\right) = \sin(1.2) = 0.93, \quad \cos(1.2) = 0.362.$$

Introducing these values in formula (66) we get,

s	e	i
100 km.	0.973 E	0.714 I
1000 km.	0.745 E	0.591 I
2000 km.	0.544 E	0.259 I

where $$I = \frac{E}{r_1}.$$

In Fig. 68 a set of curves is given showing the rise of potential over the entire length of the cable.

In Fig. 69 curves are given for the current as a function of the time for the points on the cable

$$s = 0, \quad s = 0.25\,l, \quad s = 0.5\,l, \quad s = 0.75\,l.$$

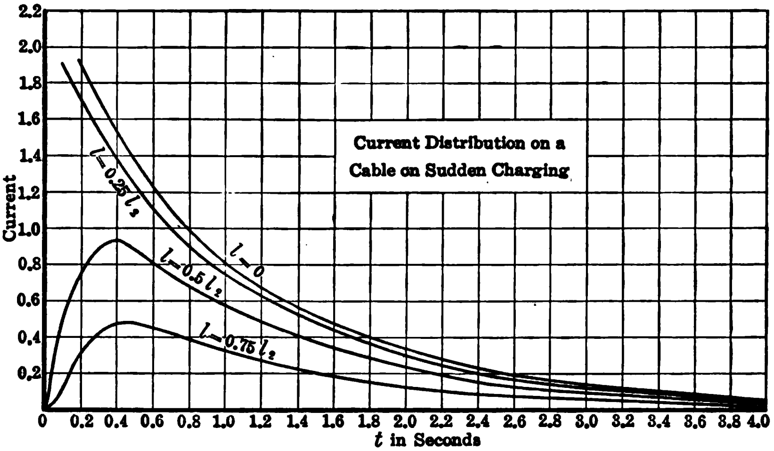

Fig. 69. — Current distribution on a cable on sudden charging.

A CHARGED CABLE HAVING A CONSTANT E.M.F. APPLIED BETWEEN ONE END OF THE CABLE AND GROUND AND SUDDENLY GROUNDED AT THE OTHER END

The permanent values of the potential and current at any point on the line distant s from the point where the e.m.f. is applied are

$$\left.\begin{aligned} E_s &= E\left(1 - \frac{s}{l}\right), \\ I_s &= \frac{E}{r_1}. \end{aligned}\right\} \qquad (68)$$

The values of the transient terms of potential and current at any point on the line are given by the following formulæ:

$$\left.\begin{aligned} e_v &= \frac{2E}{\pi}\left\{\epsilon^{-\frac{\pi^2}{r_1C_1}t}\sin\left(\frac{\pi s}{l}\right) - \frac{1}{2}\epsilon^{-\frac{4\pi^2}{r_1C_1}t}\sin\left(\frac{2\pi s}{l}\right)\right. \\ &\qquad \left. + \frac{1}{3}\epsilon^{-\frac{9\pi^2}{r_1C_1}t}\sin\left(\frac{3\pi s}{l}\right) - \cdots\right\}, \\ i_v &= -\frac{2E}{r_1}\left\{\epsilon^{-\frac{\pi^2}{r_1C_1}t}\cos\left(\frac{\pi s}{l}\right) - \epsilon^{-\frac{4\pi^2}{r_1C_1}t}\cos\left(\frac{2\pi s}{l}\right)\right. \\ &\qquad \left. + \epsilon^{\frac{9\pi^2}{r_1C_1}t}\cos\left(\frac{3\pi s}{l}\right) - \cdots\right\}. \end{aligned}\right\} \tag{69}$$

For the current at the end of the cable we may combine the second equations of (68) and (69) and put them in the following form:

$$I = \frac{E}{r_1}\left\{1 - 2\sum_{m=1}^{m=\infty}\epsilon^{-\frac{m^2\pi^2}{r_1C_1}t}\cos m\pi\right\}, \tag{70}$$

which gives the arrival curve of the current at the receiving end.

The series given by equation (70) is very slowly convergent for small values of t, and makes the calculations very laborious. Equation (70) may be put in another form which gives a rapidly convergent series for small values of t. The equivalent of formula (70) is

$$I = 2E\sqrt{\frac{C_1}{r_1\pi t}}\sum_{n=0}^{n=\infty}\epsilon^{-\frac{(2n+1)C_1r_1}{4t}}. \tag{71}$$

As an illustration we take the same data as given in the example given on page 232.

$r_1 = 6000$ ohms,
$C_1 = 40$ mf., $l = 2640$ km.
For $t = 0.001$ second, we have

$$\frac{\pi^2}{r_1C_1}t = 0.041.$$

By (70),

$$I = \frac{E}{r_1}\{1 - 2\,\epsilon^{-0.041}\cos\pi - 2\,\epsilon^{-0.164}\cos 2\pi - 2\,\epsilon^{-0.369}\cos 3\pi - \cdots\}$$
$$= \frac{E}{r_1}\{1 + 1.92 - 1.7 + 1.38 - \cdots\}.$$

It requires a very large number of terms to obtain the value of I.

By formula (71) we have

$$\frac{C_1 r_1}{4t} = \frac{40 \times 10^{-6} \times 6000}{4 \times 10^{-3}} = 60$$

and
$$I = 2E\sqrt{\frac{40 \times 10^{-6}}{6000 \times \pi \times 10^{-3}}}\,\epsilon^{-60} = E \times 2.9 \times \epsilon^{-60},$$

which shows that for the time $t = 0.001$ sec. the current at the receiving end is negligible.

If the cable is grounded all the time at one end and a constant e.m.f. is applied at the other end the permanent values of the potential and current as before will be

$$\left.\begin{aligned} E_s &= E\left(1 - \frac{s}{l}\right), \\ I_s &= \frac{E}{r_1}. \end{aligned}\right\} \qquad (72)$$

The transient terms for the potential and current are given by the formulæ

$$\left.\begin{aligned} e_v &= -\frac{2E}{\pi}\left\{\epsilon^{-\frac{\pi^2}{r_1C_1}t}\sin\left(\frac{\pi s}{l}\right) + \frac{1}{2}\epsilon^{-\frac{4\pi^2}{r_1C_1}t}\sin\left(\frac{2\pi s}{l}\right) + \cdots\right\}, \\ i_v &= \frac{2E}{r_1}\left\{\epsilon^{-\frac{\pi^2}{r_1C_1}t}\cos\left(\frac{\pi s}{l}\right) + \epsilon^{-\frac{4\pi^2}{r_1C_1}t}\cos\left(\frac{2\pi s}{l}\right) + \cdots\right\}. \end{aligned}\right\} \qquad (73)$$

CIRCUITS CONTAINING DISTRIBUTED RESISTANCE, INDUCTANCE AND CAPACITY; LINE OSCILLATIONS

We have shown in Chapter V that in a circuit containing inductance and capacity, the transient possesses an oscillatory character, unless the resistance exceeds a certain critical value depending on the inductance and capacity of the circuit. In circuits of distributed inductance and capacity, the transient is, in general, oscillatory. The discharge of an accumulated charge of atmospheric electricity on a line, a sudden change in load, an arcing ground, or a break in the circuit will set up electric oscillations on the line. Owing to the inductance and capacity of the system a flow of current stores up magnetic and dielectric energy in the system. A sudden change in circuit condition, grounding the line or a break in circuit, for instance, releases the energy stored in the system and it oscillates back and forth until the entire energy is dissipated by the resistance and leakage of the system.

Since the maximum magnetic energy must equal the maximum dielectric energy, we have

$$\left.\begin{aligned} \tfrac{1}{2} LI^2 &= \tfrac{1}{2} CE^2, \\ I &= \sqrt{\frac{C}{L}}E, \\ E &= \sqrt{\frac{L}{C}}I, \end{aligned}\right\} \tag{74}$$

$$\frac{E}{I} = \sqrt{\frac{L}{C}} = Z_0 = \frac{1}{y_0}. \tag{75}$$

The quantity Z_0 is called the natural or surge impedance, and y_0 is the natural or surge admittance.

In the case of circuits of large inductance and small capacity even moderate currents may cause a dangerous rise in voltage by a sudden change in circuit condition.

The general expressions for the oscillating current and voltage are as follows:

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\sum(P_n \cos\gamma_n s + Q_n \sin\gamma_n s)\{A_n \cos\beta_n t + B_n \sin\beta_n t\}, \\ E &= \sqrt{\frac{L}{C}}\epsilon^{-\alpha t}\sum(P_n \sin\gamma_n s + Q_n \cos\gamma_n s)\{A_n \sin\beta_n t + B_n \cos\beta_n t\}. \end{aligned}\right\} \tag{76}$$

The values of the constants $\gamma_n, \beta_n, A_n, B_n, P_n$ and Q_n depend on the terminal conditions, and

$$\alpha = \frac{1}{2}\left(\frac{R}{L} + \frac{g}{C}\right). \tag{77}$$

We shall now consider a few specific cases.

LINE GROUNDED AT ONE END AND OPEN AT THE OTHER END

The terminal conditions are

$$E = 0 \quad \text{when} \quad s = 0, \quad \text{and} \quad I = 0 \quad \text{when} \quad s = l.$$

l is the length of the line.

To satisfy these conditions we must have

$$Q_n = 0 \quad \text{and} \quad \gamma_n l = \frac{(2n-1)\pi}{2}$$

and equations (76) reduce to

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\sum_{n=1}^{n=\infty}\cos\frac{(2n-1)\pi s}{2l}\{A_n\cos\beta_n t + B_n\sin\beta_n t\},\\ E &= \sqrt{\frac{L}{C}}\,\epsilon^{-\alpha t}\sum_{n=1}^{n=\infty}\sin\frac{(2n-1)\pi s}{2l}\{A_n\sin\beta_n t - B_n\cos\beta_n t\}.\end{aligned}\right\} \qquad (78)$$

$$\beta_n = \sqrt{\frac{(2n-1)^2\pi^2}{2l^2LC} - \alpha^2}. \qquad (79)$$

If we neglect α^2 under the radical, as it is generally small compared with the term $\frac{\pi^2}{l^2LC}$, and also put $lL = L_0$ and $lC = C_0$, so that L_0 and C_0 represent the total inductance and capacity of the line, we get

$$\beta_n = \frac{(2n-1)\pi}{2\sqrt{L_0C_0}}. \qquad (80)$$

β_n is the frequency constant of the line.

The fundamental frequency of the oscillations on the line is given by

$$f_1 = \frac{\beta_1}{2\pi} = \frac{\pi}{4\pi\sqrt{L_0C_0}} = \frac{1}{4\sqrt{L_0C_0}}. \qquad (81)$$

We may therefore write

$$\beta_n = 2\pi(2n-1)f_1. \qquad (82)$$

Equations (78) may now be written in the following form:

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\sum_{n=1}^{n=\infty}\cos\frac{(2n-1)\pi s}{2l}D_n\cos\{2\pi(2n-1)f_1 t - \phi_n\},\\ E &= \sqrt{\frac{L}{C}}\,\epsilon^{-\alpha t}\sum_{n=1}^{n=\infty}\sin\frac{(2n-1)\pi s}{2l}D_n\sin\{2\pi(2n-1)f_1 t - \phi_n\}.\end{aligned}\right\} \qquad (83)$$

The current and potential on the line are complex waves consisting of fundamental waves and odd harmonics. For the fundamental wave the current is zero at the open end of the line ($s = l$) and gradually increases to its maximum value at the grounded end of the line. The potential is zero at the grounded end of the line ($s = 0$) and gradually increases to its maximum value at the open end of the line ($s = l$).

The wave length of this, the fundamental oscillation, is four times the length of the line.

The values of the constants D_n and ϕ_n depend upon the initial conditions, and have to be evaluated independently for each particular problem. In general the constants are determined in the usual manner of evaluating a Fourier Series. We may best indicate the process by the following illustration: Let the line be charged to a uniform potential E_0 but with no current in the line before the discharge. The line is then grounded at one end $s = 0$ while open at the other end $s = l$. We have then for $t = 0$, $I = 0$ for all values of s and for

$$t = 0, \quad E = E_0 \quad \text{for all values of } s.$$

From equation (78) it is evident that to satisfy the first condition we must have $A_n = 0$, and to satisfy the second condition we must have

$$E_0 = \sqrt{\frac{L}{C}}\,\epsilon^{-\alpha t}\sum_{n=1}^{n=\infty} B_n \sin\frac{(2n-1)\pi s}{2l}.$$

The values of B_n can now be determined in the usual way of evaluating the constants of a Fourier Series, that is, by multiplying both sides of the equation by $\sin\frac{(2n-1)\pi s}{2l}$ and integrating between the limits 0 and $2l$.* We shall then get for the value of the constant

$$B_n = \frac{4E_0}{(2n-1)\pi}\sqrt{\frac{C}{L}}. \tag{84}$$

Introducing the value of B_n in equation (78) we finally get

$$\left.\begin{aligned} I &= \frac{4E_0}{\pi}\sqrt{\frac{C}{L}}\,\epsilon^{-\alpha t}\left\{\cos\frac{\pi s}{2l}\sin\omega_1 t + \frac{1}{3}\cos\frac{3\pi s}{2l}\sin 3\omega_1 t \right. \\ &\qquad \left. + \frac{1}{5}\cos\frac{5\pi s}{2l}\sin 5\omega_1 t + \cdots\right\}, \\ E &= \frac{4E_0}{\pi}\epsilon^{-\alpha t}\left\{\sin\frac{\pi s}{2l}\cos\omega_1 t + \frac{1}{3}\sin\frac{3\pi s}{2l}\cos 3\omega_1 t + \cdots\right\}, \end{aligned}\right\} \tag{85}$$

where $\omega_1 = 2\pi f_1$.

LINE GROUNDED AT BOTH ENDS

Equations (76), being perfectly general, also apply in this case, but the constants γ_n and β_n are different from those discussed in the preceding section.

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* See Byerley, Fourier Series.

We have in this case the potential zero at both ends of the line, that is, $E = 0$ for $s = 0$, and $E = 0$ for $s = l$. To satisfy these conditions we must have $Q_n = 0$ and $\gamma_n l$ will have to be a multiple of π, that is

$$\gamma_n l = n\pi, \quad \gamma_n = \frac{n\pi}{l}, \tag{86}$$

and $\beta_n = \frac{n\pi}{l\sqrt{LC}} = \frac{n\pi}{\sqrt{L_0 C_0}}$, neglecting α^2 compared with $\frac{n^2\pi^2}{L_0 C_0}$.

The fundamental frequency of the oscillations is given by

$$f_1 = \frac{1}{2\sqrt{L_0 C_0}}. \tag{87}$$

Equations (76) may now be written for this case in the following form:

$$\left.\begin{aligned} E &= \sqrt{\frac{L}{C}}\,\epsilon^{-\alpha t} \sum_{n=1}^{n=\infty} D_n \sin \frac{n\pi s}{l} \sin (2\pi n f_1 t - \psi_n), \\ I &= \epsilon^{-\alpha t} \sum_{n=1}^{n=\infty} D_n \cos \frac{n\pi s}{l} \cos (2\pi n f_1 t - \psi_n). \end{aligned}\right\} \tag{88}$$

The fundamental oscillation of the current has its maximum values at either end of the line but in opposite directions; and it has zero value at the middle of the line. The fundamental oscillation of the voltage has its maximum value at the middle of the line and is zero at both ends of the line. The fundamental wave length is half the length of the line. The oscillations contain all the harmonics even and odd.

LINE OPEN AT BOTH ENDS

In this case we also make use of equations (76) as the general solution, but the terminal conditions for this problem are

$$I = 0 \quad \text{when } s = 0 \text{ and } s = l.$$

These conditions are satisfied by equations (76) if we put

$$P_n = 0 \quad \text{and} \quad \gamma_n l = n\pi \quad \text{or} \quad \gamma_n = \frac{n\pi}{l}. \tag{89}$$

The value of β_n is the same as in the previous case,

$$\beta_n = \frac{n\pi}{l\sqrt{LC}} = \frac{n\pi}{\sqrt{L_0C_0}}.$$

The fundamental frequency of the oscillation is

$$f_1 = \frac{1}{2\sqrt{L_0C_0}}.$$

Introducing the above values, equations (76) assumes the form:

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\sum_{n=1}^{n=\infty} D_n \sin\frac{n\pi s}{l}\cos\left(2\pi n f_1 t - \phi_n\right), \\ E &= -\sqrt{\frac{L}{C}}\,\epsilon^{-\alpha t}\sum_{n=1}^{n=\infty} D_n \cos\frac{n\pi s}{l}\sin\left(2\pi n f_1 t - \phi_n\right). \end{aligned}\right\} \qquad (90)$$

For the fundamental frequency the current is zero at both ends of the line and has its maximum value at the middle of the line. The potential has its maximum value at either end of the line but of opposite signs, and its value is zero at the middle of the line. The fundamental wave length is equal to half the length of the line. The oscillations on the line is of a complex form containing a fundamental wave and all of its harmonics, even as well as the odd ones.

CIRCUIT CLOSED UPON ITSELF

The terminal conditions for this case are $I_0 = I_l$ and $E_0 = E_l$, that is, the current and potential have the same values for the points $s = 0$ and $s = l$.

Both these conditions are satisfied if we put in equations(76), which is the general solution,

$$\left.\begin{aligned} &\gamma_n l = 2\,n\pi \ \text{ and } \ \gamma_n = \frac{2\,n\pi}{l}, \\ \text{and hence,}\quad &\beta_n = \frac{2\,n\pi}{l\sqrt{LC}} = \frac{2\,n\pi}{\sqrt{L_0C_0}}. \end{aligned}\right\} \qquad (91)$$

The fundamental frequency of the oscillations is

$$f_1 = \frac{\beta_1}{2\pi} = \frac{1}{\sqrt{L_0C_0}}. \qquad (92)$$

Equations (76), in their application to this case, take the forms

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\sum_{n=1}^{n=\infty}\left\{P_n \cos\frac{2\pi ns}{l} + Q_n \sin\frac{2\pi ns}{l}\right\} \\ &\qquad \{A_n \cos 2\pi nf_1 t + B_n \sin 2\pi nf_1 t\}, \\ E &= \sqrt{\frac{L}{C}}\epsilon^{-\alpha t}\sum_{n=1}^{n=\infty}\left\{P_n \sin\frac{2\pi ns}{l} + Q_n \cos\frac{2\pi ns}{l}\right\} \\ &\qquad \{A_n \sin 2\pi nf_1 t - B_n \cos 2\pi nf_1 t\}; \end{aligned}\right\} \tag{93}$$

or we may write the above equations in the following form:

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\sum_{n=1}^{n=\infty} D_n \cos\frac{2\pi ns}{l}\cos(2\pi nf_1 t - \phi_n) \\ &\qquad + F_n \sin\frac{2n\pi s}{l}\cos(2\pi nf_1 t - \phi_n), \\ E &= \sqrt{\frac{L}{C}}\epsilon^{-\alpha t}\sum_{n=1}^{n=\infty} D_n \sin\frac{2\pi ns}{l}\sin(2\pi nf_1 t - \phi_n) \\ &\qquad + F_n \cos\frac{2\pi ns}{l}\sin(2\pi nf_1 t - \phi_n). \end{aligned}\right\} \tag{94}$$

For the fundamental frequency of the oscillations equations (94) reduce to

$$\left.\begin{aligned} I &= \epsilon^{-\alpha t}\left\{D_1 \cos\frac{2\pi s}{l}\cos(\beta_1 t - \phi_1) + F_1 \sin\frac{2\pi s}{l}\cos(\beta_1 t - \phi_1)\right\}, \\ E &= \sqrt{\frac{L}{C}}\epsilon^{-\alpha t}\left\{D_1 \sin\frac{2\pi s}{l}\sin(\beta_1 t - \phi_1) + F_1 \cos\frac{2\pi s}{l}\sin(\beta_1 t - \phi_1)\right\}. \end{aligned}\right\} \tag{95}$$

TABLE XX*

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.00.$				$x = 0.02.$				$x = 0.04.$				$x = 0.06.$				$x = 0.08.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.000	0.00	0.00	0.000	1.000	0.000	0.020	0.000	1.001	0.000	0.040	0.000	0.060	0.000	1.002	0.000	1.003	0.000	0.080	0.000	0.00
0.02	1.000	0.00	0.00	0.020	1.000	0.000	0.020	0.020	1.000	0.001	0.040	0.020	1.002	0.001	0.060	0.020	1.003	0.002	0.080	0.020	0.02
0.04	1.000	0.00	0.00	0.040	1.000	0.000	0.020	0.040	1.000	0.002	0.040	0.040	1.001	0.002	0.060	0.040	1.002	0.003	0.080	0.040	0.04
0.06	0.998	0.00	0.00	0.060	0.998	0.001	0.020	0.060	0.999	0.002	0.040	0.060	1.000	0.004	0.060	0.060	1.001	0.005	0.080	0.060	0.06
0.08	0.997	0.00	0.00	0.080	0.997	0.002	0.020	0.080	0.998	0.003	0.040	0.080	0.999	0.005	0.060	0.080	1.000	0.006	0.080	0.080	0.08
0.10	0.995	0.00	0.00	0.100	0.995	0.002	0.020	0.100	0.996	0.004	0.040	0.100	0.997	0.006	0.060	0.100	0.998	0.008	0.080	0.100	0.10
0.12	0.993	0.00	0.00	0.120	0.993	0.002	0.020	0.120	0.994	0.005	0.040	0.120	0.995	0.007	0.060	0.120	0.996	0.010	0.080	0.120	0.12
0.14	0.990	0.00	0.00	0.140	0.990	0.003	0.020	0.140	0.991	0.006	0.040	0.140	0.992	0.008	0.059	0.140	0.993	0.011	0.079	0.140	0.14
0.16	0.987	0.00	0.00	0.159	0.987	0.003	0.020	0.159	0.988	0.006	0.040	0.159	0.989	0.010	0.059	0.159	0.990	0.013	0.079	0.159	0.16
0.18	0.984	0.00	0.00	0.179	0.984	0.004	0.020	0.179	0.985	0.007	0.039	0.179	0.986	0.011	0.059	0.179	0.987	0.014	0.079	0.179	0.18
0.20	0.980	0.00	0.00	0.199	0.980	0.004	0.020	0.199	0.981	0.008	0.039	0.199	0.982	0.012	0.059	0.199	0.983	0.016	0.078	0.199	0.20
0.22	0.976	0.00	0.00	0.218	0.976	0.004	0.020	0.218	0.977	0.009	0.039	0.218	0.978	0.013	0.059	0.218	0.979	0.017	0.078	0.218	0.22
0.24	0.971	0.00	0.00	0.238	0.971	0.005	0.019	0.238	0.972	0 010	0.039	0.238	0.973	0.014	0.058	0.238	0.974	0.019	0.078	0.239	0.24
0.26	0.966	0.00	0.00	0.257	0.966	0.005	0.019	0.257	0.967	0.010	0.039	0.257	0.968	0.015	0.058	0.257	0.969	0.021	0.077	0.258	0.26
0.28	0.961	0.00	0.00	0.277	0.961	0.006	0.019	0.277	0.962	0.011	0.038	0.277	0.963	0.017	0.958	0.277	0.964	0.022	0.077	0.278	0.28
0.30	0.955	0.00	0.00	0.296	0.955	0.006	0.019	0.296	0.956	0.012	0.038	0.296	0.957	0.018	0.057	0.296	0.958	0.024	0.076	0.297	0.30
0.32	0.949	0.00	0.00	0.315	0.949	0.006	0.019	0.315	0.950	0.013	0.038	0.315	0.951	0.019	0.057	0.315	0.952	0.025	0.076	0.316	0.32
0.34	0.943	0.00	0.00	0.334	0.943	0.007	0.019	0.334	0.944	0.013	0.038	0.334	0.945	0.020	0.057	0.334	0.946	0.027	0.075	0.335	0.34
0.36	0.936	0.00	0.00	0.353	0.936	0.007	0.019	0.353	0.937	0.014	0.037	0.353	0.938	0.021	0.056	0.353	0.939	0.028	0.075	0.354	0.36
0.38	0.929	0.00	0.00	0.371	0.929	0.007	0.019	0.371	0.930	0.015	0.037	0.371	0.931	0.022	0.056	0.372	0.932	0.030	0.074	0.372	0.38
0.40	0.921	0.00	0.00	0.389	0.921	0.008	0.018	0.389	0.922	0.016	0.037	0.389	0.923	0.023	0.055	0.390	0.924	0.031	0.074	0.390	0.40
0.42	0.913	0.00	0.00	0.408	0.913	0.008	0.018	0.408	0.914	0.017	0.037	0.408	0.915	0.024	0.055	0.409	0.916	0.033	0.073	0.409	0.42
0.44	0.905	0.00	0.00	0.426	0.905	0.009	0.018	0.426	0.906	0.018	0.036	0.426	0.907	0.026	0.054	0.427	0.908	0.034	0.073	0.427	0.44
0.46	0.896	0.00	0.00	0.444	0.896	0.009	0.018	0.444	0.897	0.018	0.036	0.444	0.898	0.027	0.054	0.445	0.899	0.035	0.072	0.445	0.46
0.48	0.887	0.00	0.00	0.462	0.887	0.009	0.018	0.462	0.888	0.019	0.035	0.462	0.889	0.028	0.053	0.463	0.890	0.037	0.071	0.463	0.48

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* This table is taken from a paper by W. E. Miller on Formulæ, Constants and Hyperbolic functions for Transmission Line Problems. *General Electric Review Supplement*, May, 1910.

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy)\ c + jd$

y	x = 0.00.				x = 0.02.				x = 0.04.				x = 0.06.				x = 0.08.				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	0.878	0.00	0.00	0.479	0.878	0.010	0.018	0.479	0.879	0.019	0.035	0.479	0.880	0.029	0.053	0.480	0.881	0.038	0.070	0.480	0.50
0.52	0.868	0.00	0.00	0.497	0.868	0.010	0.017	0.497	0.869	0.020	0.035	0.497	0.870	0.030	0.052	0.498	0.871	0.040	0.070	0.499	0.52
0.54	0.857	0.00	0.00	0.514	0.857	0.010	0.017	0.514	0.858	0.021	0.034	0.514	0.859	0.031	0.052	0.515	0.860	0.041	0.069	0.516	0.54
0 56	0.847	0.00	0.00	0.531	0.847	0.011	0.017	0.531	0.848	0.022	0.034	0.531	0.849	0.032	0.051	0.532	0.850	0.042	0.068	0.533	0.56
0.58	0.836	0.00	0.00	0.548	0.836	0.011	0.017	0.548	0.837	0.022	0.033	0.548	0.838	0.033	0.050	0.549	0.039	0.044	0.067	0.550	0.58
0.60	0.825	0.00	0.00	0.564	0.825	0.011	0.016	0.564	0.826	0.023	0.033	0.564	0.827	0.034	0.049	0.565	0.828	0.045	0.066	0.566	0.60
0.62	0.814	0.00	0.00	0.581	0.814	0.012	0.016	0.581	0.815	0.024	0.033	0.581	0.816	0.035	0.049	0.582	0.817	0.046	0.065	0.583	0.62
0.64	0.802	0.00	0.00	0.597	0.802	0.012	0.016	0.597	0.802	0.024	0.032	0.597	0.803	0.036	0.048	0.598	0.805	0.048	0.064	0.599	0.64
0.66	0.790	0.00	0.00	0.613	0.790	0.012	0.016	0.613	0.791	0.025	0.032	0.613	0.792	0.037	0.047	0.614	0.793	0.049	0.063	0.615	0.66
0.68	0.777	0.00	0.00	0.629	0.777	0.013	0.016	0.629	0.778	0.025	0.031	0.629	0.779	0.038	0.047	0.630	0.780	0.050	0.062	0.631	0.68
0.70	0.765	0.00	0.00	0.644	0.765	0.013	0.015	0.644	0.766	0.026	0.031	0.645	0.767	0.039	0.046	0.645	0.767	0.052	0.061	0.646	0.70
0.72	0.752	0.00	0.00	0.659	0.752	0.013	0.015	0.659	0.753	0.026	0.030	0.660	0.754	0.040	0.045	0.660	0.754	0.053	0.060	0.661	0.72
0.74	0.738	0.00	0.00	0.674	0.738	0.013	0.015	0.674	0.739	0.027	0.030	0.675	0.740	0.041	0.044	0.675	0.740	0.054	0.059	0.676	0.74
0.76	0.725	0.00	0.00	0.689	0.725	0.014	0.014	0.689	0.726	0.027	0.029	0.690	0.727	0.042	0.044	0.690	0.727	0.055	0.058	0.691	0.76
0.78	0.711	0.00	0.00	0.703	0.711	0.014	0.014	0.703	0.712	0.028	0.029	0.704	0.713	0.043	0 043	0.704	0.713	0.056	0.057	0.705	0.78
0.80	0.697	0.00	0.00	0.717	0.697	0.014	0.014	0.717	0.698	0.029	0.028	0.718	0.699	0.043	0.042	0.718	0.699	0.057	0.056	0.719	0.80
0.82	0.682	0.00	0.00	0.731	0.682	0.015	0.014	0.731	0.683	0.029	0.027	0.732	0.684	0.044	0.041	0.732	0.684	0.058	0.055	0.733	0.82
0.84	0.667	0.00	0.00	0.744	0.667	0.015	0.013	0.744	0.667	0.030	0.027	0.745	0.688	0.045	0.040	0.745	0.669	0.060	0.053	0.746	0.84
0.86	0.652	0.00	0.00	0.758	0.652	0.015	0.013	0.758	0.652	0.030	0.026	0.759	0.653	0.046	0.039	0.759	0.654	0.061	0.052	0.760	0.86
0.88	0.637	0.00	0 00	0.771	0.637	0.015	0.013	0.771	0.637	0.031	0.025	0.772	0.638	0.046	0.038	0.772	0.639	0.062	0.051	0.773	0.88
0.90	0.622	0.00	0.00	0.783	0.622	0.016	0.012	0.783	0.622	0.031	0.025	0.784	0.623	0.047	0.037	0.784	0.624	0.062	0.050	0.786	0.90
0.92	0.606	0.00	0.00	0.796	0.606	0.016	0.012	0.796	0.606	0.032	0.024	0.797	0.607	0.048	0.036	0.797	0.608	0.064	0.049	0.799	0.92
0.94	0.590	0.00	0.00	0.808	0.590	0.016	0.012	0.808	0.590	0.032	0.024	0.809	0.591	0.048	0.035	0.809	0.592	0.065	0.047	0.811	0.94
0.96	0.574	0.00	0.00	0.819	0.574	0.016	0.011	0.819	0.574	0.033	0.023	0.820	0.575	0.049	0.034	0.820	0.576	0.066	0.046	0.822	0.96
0.98	0.557	0.00	0.00	0.830	0.557	0.017	0.011	0.830	0.557	0.033	0.023	0.831	0.558	0.050	0.033	0.831	0.559	0.066	0.044	0.833	0.98
1.00	0.540	0.00	0.00	0.841	0.540	0.017	0.011	0.841	0.540	0.034	0.022	0.842	0.541	0.050	0.032	0.843	0.542	0.067	0.043	0.844	1.00

TABLE XX — (*Continued*)

VALUES OF COSH $(x+jy) = a+jb$ AND SINH $(x+jy)\ c+jd$

y	$x = 0.10.$				$x = 0.12.$				$x = 0.14.$				$x = 0.16.$				$x = 0.18.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.005	0.000	0.100	0.000	1.007	0.000	0.120	0.000	1.010	0.000	0.140	0.000	1.013	0.000	0.161	0.000	1.016	0.000	0.181	0.000	0.00
0.02	1.005	0.002	0.100	0.020	1.007	0.002	0.120	0.020	1.010	0.003	0.140	0.020	1.013	0.003	0.161	0.020	1.016	0.003	0.181	0.020	0.02
0.04	1.004	0.004	0.100	0.040	1.006	0.005	0.120	0.040	1.009	0.006	0.140	0.040	1.012	0.006	0.160	0.040	1.015	0.007	0.180	0.041	0.04
0.06	1.003	0.006	0.100	0.060	1.005	0.007	0.120	0.060	1.008	0.008	0.140	0.061	1.011	0.010	0.160	0.061	1.014	0.011	0.180	0.061	0.06
0.08	1.002	0.008	0.100	0.080	1.004	0.010	0.120	0.080	1.007	0.011	0.140	0.081	1.010	0.013	0.160	0.081	1.013	0.014	0.180	0.081	0.08
0.10	1.000	0.010	0.100	0.100	1.002	0.012	0.120	0.101	1.005	0.014	0.140	0.101	1.008	0.016	0.160	0.101	1.011	0.018	0.180	0.102	0.10
0.12	0.998	0.012	0.099	0.121	1.000	0.014	0.120	0.121	1.003	0.017	0.139	0.121	1.006	0.019	0.160	0.122	1.009	0.022	0.180	0.122	0.12
0.14	0.995	0.014	0.099	0.141	0.997	0.017	0 119	0.141	1.000	0.020	0.139	0.141	1.003	0.022	0.159	0.142	1.006	0.025	0.179	0.142	0.14
0.16	0.992	0.016	0.099	0.160	0.994	0.019	0.119	0.160	0.997	0.022	0.138	0.161	1.000	0.025	0.159	0.161	1.003	0.029	0.178	0.162	0.16
0.18	0.989	0.018	0.098	0.180	0.991	0.022	0.119	0.180	0.994	0.025	0.138	0.181	0.997	0.029	0.158	0.181	1.000	0.032	0.178	0.182	0.18
0.20	0.985	0.020	0.098	0.200	0.987	0.024	0.118	0.200	0.990	0.028	0.138	0.201	0.992	0.032	0.158	0.202	0.996	0.036	0.177	0.202	0.20
0.22	0.981	0.022	0 098	0.219	0.983	0.026	0.118	0.220	0.986	0.031	0.137	0.220	0.988	0.035	0.157	0.221	0.992	0.039	0.177	0.221	0.22
0.24	0.976	0.024	0.097	0.239	0.978	0.028	0.117	0.240	0.981	0.033	0.136	0.240	0.983	0.038	0.156	0.241	0.987	0.043	0.176	0.242	0.24
0.26	0.971	0.026	0.097	0.258	0.973	0.031	0.116	0.259	0.975	0.036	0.136	0.259	0.978	0.041	0.156	0.260	0.981	0.046	0.175	0.261	0.26
0.28	0.966	0.028	0.096	0.278	0.968	0.033	0 116	0.279	0.970	0.039	0.135	0.280	0.973	0.044	0.155	0.281	0.976	0.050	0.174	0.281	0.28
0.30	0.960	0.030	0.095	0.297	0.962	0.036	0.115	0.298	0.964	0.041	0.134	0.299	0.967	0.048	0.154	0.299	0.971	0.053	0.173	0.301	0.30
0.32	0.954	0.031	0.095	0.316	0.956	0.038	0.115	0.317	0.958	0.044	0.133	0.318	0.961	0.051	0.152	0.319	0.964	0.057	0.172	0.320	0.32
0.34	0.948	0.033	0.094	0.335	0.950	0.040	0.114	0.336	0.952	0.047	0.132	0.337	0.955	0.054	0.151	0.338	0.958	0.060	0.171	0.339	0.34
0.36	0.941	0.035	0.094	0.354	0.943	0.042	0.113	0.356	0.945	0.049	0.131	0.356	0.948	0.057	0.150	0.357	0.951	0.064	0.169	0.358	0.36
0.38	0.934	0.037	0.093	0.373	0.936	0.045	0.112	0.374	0.938	0.052	0.130	0.375	0.941	0.059	0.149	0.376	0.944	0.067	0.168	0.377	0.38
0.40	0.926	0.039	0.092	0.391	0.928	0.047	0.111	0.392	0.930	0.055	0.129	0.393	0.933	0.062	0.148	0.394	0.936	0.070	0.166	0.395	0.40
0.42	0.918	0.041	0.091	0.410	0.920	0.049	0.110	0.411	0.922	0.057	0.128	0.412	0.925	0.065	0.147	0.413	0.928	0.073	0.165	0.414	0.42
0.44	0.910	0.043	0.091	0.428	0.912	0.051	0.109	0.429	0.914	0.060	0.127	0.430	0.916	0.068	0.146	0.431	0.919	0.077	0.164	0.433	0.44
0.46	0.901	0.044	0.090	0.446	0.902	0.053	0.108	0.447	0.904	0.062	0.126	0.448	0.907	0.071	0.144	0.449	0.910	0.080	0.162	0.451	0.46
0.48	0.891	0.046	0.089	0.464	0.893	0.055	0.107	0.465	0.895	0.065	0.125	0.467	0.898	0.074	0.142	0.468	0.901	0.083	0.160	0.469	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jh) = c + jy$

y	$x = 0.10.$				$x = 0.12.$				$x = 0.14.$				$x = 0.16.$				$x = 0.18.$				y
	a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	
0.50	0.882	0.048	0.088	0.482	0.884	0.057	0.106	0.482	0.887	0.067	0.123	0.484	0.889	0.077	0.141	0.485	0.892	0.087	0.158	0.487	0.50
0.52	0.872	0.050	0.087	0.500	0.874	0.060	0.104	0.501	0.877	0.069	0.122	0.502	0.879	0.080	0.140	0.503	0.882	0.090	0.157	0.505	0.52
0.54	0.861	0.051	0.086	0.517	0.863	0.062	0.103	0.518	0.865	0.072	0.121	0.519	0.868	0.082	0.138	0.520	0.871	0.093	0.155	0.522	0.54
0.56	0.851	0.053	0.085	0.534	0.853	0.064	0.102	0.535	0.855	0.074	0.119	0.536	0.858	0.085	0.136	0.537	0.861	0.096	0.153	0.540	0.56
0.58	0.840	0.055	0.084	0.551	0.842	0.066	0.100	0.553	0.844	0.077	0.118	0.554	0.847	0.087	0.134	0.555	0.850	0.099	0.151	0.557	0.58
0.60	0.829	0.056	0.082	0.567	0.831	0.068	0.099	0.568	0.833	0.079	0.116	0.570	0.836	0.091	0.132	0.571	0.838	0.102	0.149	0.573	0.60
0.62	0.818	0.058	0.081	0.584	0.820	0.070	0 098	0.585	0.822	0.081	0.114	0.587	0.824	0.093	0.130	0.588	0.827	0.105	0.147	0.590	0.62
0.64	0.806	0.060	0.080	0.600	0.808	0.072	0.097	0.602	0.810	0.083	0.112	0.603	0.812	0.096	0.128	0.604	0.815	0.108	0.145	0.607	0.64
0.66	0.794	0.061	0.079	0.616	0.796	0.074	0.095	0.618	0.798	0.086	0.110	0.619	0.800	0.098	0.127	0.620	0.803	0.111	0.143	0.623	0.66
0.68	0.781	0.063	0.078	0.632	0.783	0.076	0.094	0.634	0.785	0.088	0.109	0.635	0.787	0.101	0.125	0.636	0.790	0.114	0.140	0.639	0.68
0.70	0.769	0.064	0.076	0.647	0.771	0.078	0.092	0.649	0.773	0.090	0.108	0.650	0.775	0.103	0.123	0.652	0.777	0.116	0.138	0.654	0.70
0.72	0.756	0.066	0.075	0.662	0.758	0.079	0.091	0.664	0.759	0.092	0.106	0.665	0.762	0.105	0.121	0.667	0.764	0.118	0.136	0.670	0.72
0.74	0.742	0.067	0.074	0.677	0.743	0.081	0.089	0.679	0.745	0.094	0.104	0.681	0.748	0.108	0.119	0.683	0.750	0.122	0.134	0.685	0.74
0.76	0.729	0.069	0.072	0.602	0.730	0.083	0.087	0.694	0.732	0.096	0.102	0.696	0.734	0.111	0.117	0.698	0.737	0.124	0.131	0.700	0.76
0.78	0.714	0.070	0.071	0.706	0.716	0.085	0.086	0.708	0.718	0.099	0.100	0.710	0.720	0.113	0.115	0.712	0.722	0.127	0.128	0.714	0.78
0.80	0.700	0.072	0.070	0.720	0.702	0.086	0.084	0.722	0.704	0.101	0.098	0.724	0.706	0.115	0.112	0.726	0.708	0.129	0.126	0.729	0.80
0.82	0.685	0.073	0.068	0.734	0.687	0.088	0.082	0.736	0.689	0.102	0.096	0.738	0.691	0.118	0.110	0.740	0.693	0.132	0.123	0.743	0.82
0.84	0.670	0.074	0.067	0.747	0.672	0.090	0.080	0.749	0.674	0.104	0.094	0.751	0.676	0.120	0.107	0.753	0.678	0.134	0.120	0.756	0.84
0 86	0.655	0.076	0.065	0.761	0.657	0.091	0.079	0.763	0.658	0.106	0.091	0.765	0.660	0.122	0.105	0.768	0.662	0.137	0.118	0.770	0.86
0.88	0.640	0.077	0.064	0.775	0.642	0.093	0.077	0.777	0.643	0.108	0.089	0.779	0.645	0.124	0.102	0.781	0.647	0.139	0.115	0.783	0.88
0.90	0.625	0 078	0.062	0.787	0.626	0.094	0.075	0.789	0.628	0.110	0.087	0.791	0.630	0.126	0.100	0.793	0.632	0.141	0.112	0.796	0.90
0.92	0.609	0.080	0.061	0.800	0.610	0.096	0.073	0.802	0.612	0.112	0.085	0.804	0.614	0.128	0.098	0.806	0.616	0.143	0.109	0.809	0.92
0.94	0.593	0.081	0.059	0.812	0.594	0.097	0.071	0.814	0.596	0.113	0.083	0.816	0.598	0.130	0.095	0.818	0.600	0.146	0.106	0.821	0.94
0.96	0.577	0.082	0.057	0.823	0.578	0.098	0.069	0.825	0.580	0.115	0.081	0.827	0.581	0.132	0.093	0.829	0.583	0.148	0.104	0.832	0.96
0.98	0.560	0.083	0.056	0.834	0.561	0.100	0.067	0 836	0.563	0.116	0.078	0.838	0.564	0.134	0.090	0.841	0.566	0.150	0.101	0.843	0.98
1.00	0.543	0.084	0.054	0.846	0.544	0.101	0.065	0.847	0.545	0.118	0.076	0.849	0.547	0.135	0.087	0.852	0.549	0.152	0.098	0.855	1.00

TABLE XX—(*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.20.$				$x = 0.22.$				$x = 0.24.$				$x = 0.26.$				$x = 0.28.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.020	0.000	0.201	0.000	1.024	0.000	0.221	0.000	1.029	0.000	0.242	0.000	1.034	0.000	0.263	0.000	1.039	0.000	0.284	0.000	0.00
0.02	1.020	0.004	0.201	0.020	1.024	0.004	0.221	0.020	1.029	0.005	0.242	0.021	1.034	0.005	0.263	0.021	1.039	0.005	0.284	0.021	0.02
0.04	1.019	0.008	0.201	0.041	1.023	0.009	0.221	0.041	1.028	0.010	0.242	0.041	1.033	0.010	0.263	0.041	1.038	0.011	0.284	0.042	0.04
0.06	1.018	0.012	0.201	0.061	1.022	0.013	0.220	0.061	1.027	0.014	0.241	0.062	1.032	0.016	0.262	0.062	1.037	0.017	0.283	0.062	0.06
0.08	1.017	0.016	0.200	0.082	1.021	0.018	0.220	0.082	1.026	0.019	0.241	0.082	1.031	0.021	0.262	0.083	1.036	0.023	0.283	0.083	0.08
0.10	1.015	0.020	0.200	0.102	1.019	0.022	0.220	0.102	1.024	0.024	0.240	0.103	1.029	0.026	0.261	0.103	1.034	0.028	0.282	0.104	0.10
0.12	1.013	0.024	0.200	0.122	1.017	0.027	0.220	0.123	1.022	0.029	0.240	0.123	1.027	0.031	0.261	0.124	1.032	0.034	0.281	0.125	0.12
0.14	1.010	0.028	0.199	0.143	1.014	0.031	0.220	0.143	1.019	0.034	0.239	0.144	1.024	0.037	0.260	0.145	1.029	0.040	0.280	0.146	0.14
0.16	1.007	0.032	0.199	0.162	1.011	0.035	0.219	0.163	1.015	0.038	0.239	0.164	1.021	0.042	0.259	0.165	1.026	0.045	0.279	0.165	0.16
0.18	1.004	0.036	0.198	0.183	1.008	0.040	0.218	0.183	1.012	0.043	0.238	0.184	1.017	0.047	0.258	0.185	1.023	0.051	0.278	0.186	0.18
0.20	1.000	0.040	0.197	0.203	1.004	0.044	0.217	0.204	1.008	0.048	0.237	0.205	1.013	0.052	0.257	0.206	1.019	0.057	0.277	0.206	0.20
0.22	0.996	0.044	0.197	0.222	1.000	0.048	0.216	0.223	1.004	0.053	0.236	0.224	1.009	0.058	0.256	0.225	1.014	0.062	0.276	0.226	0.22
0.24	0.990	0.048	0.196	0.243	0.995	0.052	0.215	0.244	0.999	0.058	0.235	0.245	1.004	0.064	0.255	0.246	1.009	0.067	0.275	0.247	0.24
0.26	0.985	0.052	0.195	0.262	0.989	0.057	0.214	0.263	0.994	0.062	0.234	0.264	0.999	0.067	0.254	0.265	1.004	0.073	0.274	0.267	0.26
0.28	0.980	0.056	0.194	0.282	0.984	0.061	0.213	0.284	0.989	0.066	0.232	0.285	0.994	0.072	0.252	0.286	0.999	0.078	0.272	0.288	0.28
0.30	0.974	0.059	0.192	0.301	0.978	0.066	0.212	0.303	0.983	0.071	0.231	0.305	0.987	0.077	0.251	0.306	0.993	0.084	0.270	0.308	0.30
0.32	0.968	0.063	0.191	0.321	0.972	0.070	0.211	0.323	0.976	0.076	0.230	0.324	0.981	0.083	0.250	0.325	0.986	0.089	0.269	0.327	0.32
0.34	0.962	0.067	0.190	0.341	0.966	0.074	0.209	0.342	0.970	0.081	0.228	0.344	0.975	0.088	0.248	0.345	0.980	0.094	0.267	0.347	0.34
0.36	0.955	0.071	0.188	0.360	0.959	0.078	0.207	0.362	0.963	0.085	0.226	0.363	0.968	0.093	0.246	0.365	0.973	0.100	0.265	0.367	0.36
0.38	0.948	0.075	0.187	0.378	0.952	0.082	0.206	0.380	0.956	0.090	0.225	0.382	0.961	0.097	0.244	0.384	0.966	0.105	0.263	0.386	0.38
0.40	0.939	0.078	0.185	0.397	0.943	0.086	0.205	0.398	0.948	0.094	0.223	0.400	0.952	0.102	0.242	0.402	0.957	0.110	0.261	0.404	0.40
0.42	0.931	0.082	0.184	0.416	0.935	0.090	0.203	0.418	0.939	0.099	0.221	0.420	0.944	0.107	0.240	0.422	0.949	0.115	0.258	0.424	0.42
0.44	0.923	0.086	0.182	0.435	0.927	0.094	0.201	0.436	0.931	0.103	0.219	0.438	0.935	0.112	0.238	0.440	0.941	0.120	0.256	0.443	0.44
0.46	0.914	0.089	0.180	0.453	0.918	0.098	0.199	0.455	0.922	0.107	0.216	0.457	0.926	0.116	0.235	0.459	0.931	0.126	0.254	0.461	0.46
0.48	0.905	0.093	0.178	0.471	0.909	0.102	0.197	0.473	0.913	0.112	0.214	0.475	0.917	0.121	0.233	0.477	0.922	0.131	0.251	0.480	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.20.$				$x = 0.22.$				$x = 0.24.$				$x = 0.26.$				$x = 0.28.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	0.895	0.096	0.177	0.489	0.899	0.106	0.195	0.492	0.903	0.116	0.212	0.493	0.908	0.126	0.230	0.495	0.913	0.136	0.249	0.498	0.50
0.52	0.885	0.100	0.175	0.507	0.889	0.110	0.193	0.509	0.893	0.120	0.209	0.511	0.897	0.130	0.228	0.513	0.902	0.141	0.246	0.517	0.52
0.54	0.874	0.103	0.172	0.524	0.878	0.114	0.190	0.526	0.882	0.125	0.207	0.528	0.886	0.135	0.225	0.530	0.891	0.146	0.242	0.534	0.54
0.56	0.864	0.106	0.170	0.542	0.868	0.118	0.188	0.544	0.872	0.128	0.205	0.546	0.876	0.140	0.223	0.549	0.881	0.150	0.239	0.552	0.56
0.58	0.853	0.110	0.168	0.559	0.856	0.122	0.185	0.561	0.860	0.132	0.202	0.563	0.864	0.144	0.219	0.566	0.870	0.155	0.236	0.570	0.58
0.60	0.842	0.114	0.166	0.576	0.845	0.125	0.183	0.578	0.849	0.137	0.199	0.580	0.851	0.148	0.216	0.583	0.858	0.160	0.233	0.586	0.60
0.62	0.830	0.117	0.164	0.593	0.834	0.129	0.181	0.595	0.838	0.141	0.197	0.598	0.842	0.152	0.213	0.601	0.846	0.164	0.231	0.604	0.62
0.64	0.818	0.120	0.162	0.609	0.821	0.132	0.178	0.611	0.825	0.144	0.194	0.614	0.829	0.157	0.210	0.617	0.833	0.169	0.227	0.620	0.64
0.66	0.806	0.123	0.159	0.625	0.809	0.136	0.175	0.628	0.813	0.148	0.191	0.631	0.817	0.161	0.207	0.634	0.821	0.174	0.224	0.637	0.66
0.68	0.793	0.126	0.156	0.642	0.796	0.140	0.172	0.644	0.799	0.152	0.188	0.647	0.803	0.166	0.204	0.650	0.808	0.178	0.220	0.654	0.68
0.70	0.780	0.130	0.154	0.657	0.784	0.143	0.170	0.660	0.787	0.156	0.185	0.663	0.791	0.170	0.201	0.666	0.795	0.182	0.217	0.669	0.70
0.72	0.767	0.132	0.152	0.672	0.770	0.146	0.167	0.675	0.773	0.160	0.182	0.678	0.778	0.173	0.198	0.681	0.782	0.186	0.213	0.685	0.72
0.74	0.753	0.135	0.149	0.687	0.756	0.150	0.164	0.690	0.759	0.163	0.179	0.693	0.763	0.177	0.194	0.696	0.768	0.190	0.209	0.701	0.74
0.76	0.740	0.138	0.146	0.703	0.743	0.153	0.161	0.706	0.746	0.166	0.175	0.709	0.750	0.181	0.191	0.712	0.754	0.195	0.205	0.716	0.76
0.78	0.725	0.141	0.143	0.717	0.728	0.156	0.158	0.720	0.731	0.170	0.172	0.723	0.735	0.185	0.187	0.726	0.739	0.200	0.201	0.731	0.78
0.80	0.711	0.144	0.140	0.732	0.714	0.159	0.155	0.734	0.717	0.173	0.169	0.738	0.721	0.188	0.183	0.741	0.724	0.203	0.197	0.745	0.80
0.82	0.696	0.147	0.137	0.745	0.699	0.162	0.152	0.749	0.702	0.176	0.165	0.752	0.705	0.192	0.179	0.755	0.708	0.206	0.193	0.760	0.82
0.84	0.680	0.150	0.134	0.758	0.683	0.165	0.148	0.762	0.686	0.180	0.161	0.765	0.690	0.195	0.175	0.768	0.693	0.210	0.189	0.773	0.84
0.86	0.665	0.152	0.131	0.773	0.668	0.168	0.144	0.776	0.671	0.183	0.158	0.780	0.674	0.199	0.171	0.783	0.677	0.214	0.185	0.788	0.86
0.88	0.650	0.155	0.128	0.786	0.653	0.171	0.141	0.790	0.655	0.186	0.154	0.794	0.659	0.203	0.167	0.798	0.662	0.218	0.180	0.802	0.88
0.90	0.634	0.158	0.125	0.799	0.637	0.174	0.138	0.802	0.640	0.190	0.151	0.806	0.643	0.206	0.162	0.810	0.647	0.222	0.176	0.814	0.90
0.92	0.618	0.160	0.122	0.812	0.621	0.177	0.134	0.815	0.624	0.193	0.147	0.819	0.627	0.209	0.159	0.823	0.630	0.225	0.171	0.827	0.92
0.94	0.602	0.162	0.119	0.824	0.604	0.179	0.131	0.828	0.607	0.195	0.143	0.832	0.610	0.212	0.155	0.836	0.613	0.228	0.167	0.840	0.94
0.96	0.586	0.164	0.115	0.835	0.588	0.182	0.127	0.839	0.591	0.197	0.139	0.843	0.594	0.215	0.151	0.847	0.597	0.231	0.163	0.851	0.96
0.98	0.568	0.167	0.112	0.847	0.571	0.184	0.123	0.850	0.573	0.200	0.135	0.854	0.576	0.218	0.146	0.858	0.579	0.235	0.158	0.863	0.98
1.00	0.551	0.169	0.109	0.858	0.553	0.186	0.120	0.861	0.556	0.203	0.131	0.865	0.558	0.221	0.142	0.870	0.561	0.238	0.153	0.874	1.00

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.30.$				$x = 0.32.$				$x = 0.34.$				$x = 0.36.$				$x = 0.38.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.045	0.000	0.304	0.000	1.052	0.000	0.325	0.000	1.058	0.000	0.347	0.000	1.065	0.000	0.368	0.000	1.073	0.000	0.389	0.000	0.00
0.02	1.045	0.006	0.304	0.021	1.051	0.006	0.325	0.021	1.058	0.007	0.346	0.021	1.065	0.007	0.367	0.021	1.073	0.008	0.389	0.021	0.02
0.04	1.044	0.012	0.304	0.042	1.051	0.013	0.325	0.042	1.057	0.014	0.346	0.042	1.064	0.015	0.367	0.043	1.072	0.016	0.389	0.043	0.04
0.06	1.043	0.018	0.303	0.063	1.050	0.019	0.324	0.063	1.056	0.021	0.345	0.064	1.063	0.022	0.366	0.064	1.071	0.023	0.338	0.064	0.06
0.08	1.042	0.024	0.303	0.084	1.048	0.026	0.324	0.084	1.055	0.028	0.345	0.085	1.062	0.029	0.366	0.085	1.070	0.031	0.387	0.086	0.08
0.10	1.040	0.030	0.303	0.104	1.046	0.032	0.324	0.105	1.053	0.035	0.344	0.106	1.060	0.037	0.366	0.107	1.068	0.039	0.387	0.107	0.10
0.12	1.038	0.036	0.302	0.125	1.044	0.039	0.323	0.126	1.051	0.041	0.344	0.127	1.058	0.044	0.365	0.128	1.066	0.047	0.386	0.129	0.12
0.14	1.035	0.042	0.301	0.146	1.041	0.045	0.322	0.147	1.048	0.048	0.343	0.148	1.055	0.051	0.364	0.149	1.062	0.054	0.385	0.150	0.14
0.16	1.032	0.048	0.300	0.166	1.038	0.052	0.321	0.167	1.045	0.055	0.342	0.168	1.052	0.059	0.363	0.169	1.059	0.062	0.384	0.171	0.16
0.18	1.028	0.055	0.299	0.187	1.035	0.058	0.320	0.188	1.042	0.062	0.341	0.189	1.048	0.066	0.362	0.191	1.056	0.070	0.382	0.192	0.18
0.20	1.024	0.061	0.298	0.208	1.031	0.065	0.319	0.209	1.037	0.069	0.340	0.211	1.044	0.074	0.360	0.212	1.052	0.078	0.380	0.214	0.20
0.22	1.019	0.066	0.297	0.228	1.026	0.071	0.317	0.229	1.033	0.075	0.338	0.231	1.039	0.080	0.358	0.232	1.047	0.085	0.378	0.234	0.22
0.24	1.014	0.072	0.295	0.249	1.021	0.077	0.315	0.251	1.028	0.082	0.336	0.253	1.034	0.087	0.356	0.254	1.042	0.093	0.377	0.255	0.24
0.26	1.009	0.078	0.294	0.269	1.016	0.083	0.314	0.271	1.022	0.089	0.334	0.273	1.029	0.095	0.354	0.274	1.037	0.100	0.375	0.276	0.26
0.28	1.004	0.084	0.293	0.289	1.011	0.090	0.312	0.292	1.017	0.096	0.332	0.294	1.024	0.102	0.353	0.295	1.031	0.108	0.373	0.297	0.28
0.30	0.999	0.090	0.291	0.309	1.004	0.096	0.310	0.311	1.011	0.102	0.331	0.313	1.017	0.108	0.351	0.314	1.025	0.115	0.371	0.318	0.30
0.32	0.993	0.096	0.289	0.329	0.998	0.102	0.308	0.331	1.004	0.109	0.329	0.333	1.011	0.116	0.349	0.334	1.018	0.122	0.369	0.338	0.32
0.34	0.987	0.102	0.287	0.349	0.992	0.108	0.306	0.351	0.998	0.116	0.327	0.353	1.004	0.123	0.347	0.355	1.012	0.130	0.366	0.358	0.34
0.36	0.980	0.107	0.285	0.369	0.984	0.115	0.304	0.371	0.991	0.122	0.324	0.373	0.997	0.130	0.343	0.375	1.004	0.138	0.363	0.379	0.36
0.38	0.972	0.113	0.283	0.388	0.977	0.121	0.302	0.390	0.983	0.128	0.321	0.393	0.990	0.136	0.341	0.395	0.997	0.145	0.361	0.398	0.38
0.40	0.963	0.118	0.280	0.407	0.968	0.126	0.300	0.409	0.975	0.134	0.319	0.412	0.981	0.143	0.339	0.414	0.988	0.151	0.358	0.417	0.40
0.42	0.954	0.124	0.277	0.426	0.960	0.133	0.297	0.428	0.966	0.141	0.317	0.431	0.973	0.150	0.336	0.435	0.980	0.158	0.356	0.438	0.42
0.44	0.945	0.130	0.275	0.445	0.952	0.138	0.294	0.447	0.958	0.148	0.313	0.450	0.964	0.156	0.333	0.453	0.971	0.166	0.352	0.457	0.44
0.46	0.936	0.135	0.273	0.464	0.942	0.144	0.291	0.466	0.948	0.154	0.310	0.469	0.955	0.163	0.329	0.473	0.961	0.172	0.348	0.476	0.46
0.48	0.928	0.140	0.270	0.483	0.933	0.150	0.288	0.485	0.939	0.160	0.307	0.488	0.945	0.170	0.325	0.492	0.952	0.179	0.344	0.495	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.30.$				$x = 0.32.$				$x = 0.34.$				$x = 0.36.$				$x = 0.38.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	0.917	0.146	0.267	0.501	0.923	0.156	0.285	0.504	0.929	0.166	0.304	0.507	0.936	0.176	0.322	0.510	0.942	0.186	0.341	0.514	0.50
0.52	0.908	0.151	0.263	0.519	0.913	0.162	0.282	0.522	0.919	0.172	0.300	0.525	0.925	0.182	0.319	0.530	0.932	0.193	0.338	0.533	0.52
0.54	0.898	0.156	0.260	0.537	0.901	0.167	0.278	0.540	0.907	0.178	0.296	0.543	0.913	0.189	0.315	0.548	0.920	0.200	0.334	0.552	0.54
0.56	0.887	0.161	0.257	0.555	0.891	0.172	0.275	0.558	0.896	0.184	0.293	0.561	0.903	0.195	0.311	0.566	0.909	0.206	0.329	0.570	0.56
0.58	0.875	0.166	0.254	0.573	0.879	0.178	0.272	0.576	0.885	0.190	0.290	0.580	0.891	0.201	0.307	0.584	0.897	0.212	0.325	0.588	0.58
0.60	0.863	0.172	0.251	0.599	0.867	0.183	0.268	0.593	0.873	0.195	0.286	0.597	0.879	0.207	0.303	0.601	0.885	0.219	0.321	0.605	0.60
0.62	0.851	0.177	0.248	0.607	0.856	0.189	0.264	0.611	0.862	0.201	0.282	0.615	0.867	0.214	0.299	0.619	0.874	0.226	0.317	0.623	0.62
0.64	0.839	0.181	0.244	0.624	0.843	0.194	0.260	0.628	0.849	0.207	0.279	0.632	0.854	0.219	0.295	0.636	0.861	0.233	0.312	0.641	0.64
0.66	0.826	0.186	0.240	0.641	0.831	0.199	0.256	0.645	0.836	0.212	0.274	0.649	0.842	0.225	0.290	0.653	0.848	0.239	0.307	0.658	0.66
0.68	0.813	0.191	0.236	0.657	0.817	0.204	0.252	0.661	0.822	0.218	0.269	0.666	0.828	0.231	0.285	0.670	0.834	0.245	0.302	0.675	0.68
0.70	0.800	0.196	0.233	0.673	0.804	0.209	0.249	0.677	0.810	0.223	0.265	0.682	0.815	0.237	0.283	0.686	0.821	0.251	0.297	0.691	0.70
0.72	0.787	0.200	0.229	0.689	0.791	0.214	0.244	0.693	0.796	0.228	0.261	0.698	0.801	0.242	0.276	0.702	0.807	0.256	0.293	0.707	0.72
0.74	0.773	0.205	0.224	0.705	0.776	0.219	0.240	0.709	0.781	0.234	0.255	0.714	0.786	0.248	0.271	0.718	0.791	0.262	0.288	0.723	0.74
0.76	0.758	0.210	0.221	0.720	0.762	0.224	0.236	0.724	0.767	0.239	0.251	0.729	0.772	0.254	0.267	0.734	0.777	0.268	0.282	0.739	0.76
0.78	0.744	0.214	0.216	0.735	0.748	0.229	0.231	0.739	0.753	0.243	0.246	0.744	0.758	0.259	0.261	0.749	0.763	0.274	0.276	0.754	0.78
0.80	0.728	0.218	0.212	0.750	0.733	0.233	0.226	0.754	0.738	0.248	0.242	0.759	0.743	0.264	0.256	0.764	0.748	0.279	0.271	0.769	0.80
0.82	0.714	0.222	0.207	0.764	0.717	0.237	0.222	0.769	0.722	0.254	0.236	0.774	0.727	0.269	0.251	0.779	0.732	0.284	0.265	0.784	0.82
0.84	0.697	0.227	0.202	0.778	0.701	0.242	0.217	0.782	0.706	0.257	0.231	0.787	0.711	0.274	0.245	0.793	0.716	0.289	0.259	0.798	0.84
0.86	0.682	0.230	0.198	0.792	0.686	0.246	0.212	0.797	0.690	0.263	0.226	0.802	0.695	0.279	0.240	0.808	0.700	0.295	0.254	0.814	0.86
0.88	0.666	0.235	0.194	0.806	0.670	0.250	0.207	0.811	0.674	0.266	0.220	0.817	0.679	0.284	0.234	0.821	0.684	0.300	0.248	0.827	0.88
0.90	0.650	0.238	0.189	0.819	0.654	0.255	0.202	0.823	0.658	0.271	0.215	0.829	0.663	0.288	0.229	0.834	0.668	0.305	0.242	0.840	0.90
0.92	0.634	0.242	0.184	0.832	0.637	0.259	0.197	0.837	0.641	0.275	0.210	0.843	0.646	0.293	0.223	0.848	0.651	0.309	0.235	0.854	0.92
0.94	0.617	0.245	0.179	0.845	0.620	0.262	0.192	0.849	0.625	0.280	0.204	0.855	0.629	0.297	0.216	0.861	0.634	0.314	0.229	0.867	0.94
0.96	0.600	0.249	0.175	0.856	0.604	0.266	0.186	0.861	0.608	0.284	0.199	0.867	0.612	0.300	0.211	0.873	0.617	0.319	0.223	0.879	0.96
0.98	0.582	0.252	0.169	0.868	0.586	0.270	0.181	0.873	0.589	0.287	0.193	0.879	0.593	0.305	0.205	0.884	0.598	0.323	0.216	0.891	0.98
1.00	0.565	0.256	0.164	0.880	0.568	0.274	0.176	0.884	0.572	0.291	0.187	0.890	0.575	0.310	0.199	0.896	0.580	0.327	0.210	0.903	1.00

TABLE XX — (*Continued*)

VALUES OF COSH $(x+jy)=a+jb$ AND SINH $(x+jy)=c+jd$

y	$x = 0.40.$				$x = 0.42.$				$x = 0.44.$				$x = 0.46.$				$x = 0.48.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.081	0.000	0.411	0.000	1.089	0.000	0.432	0.000	1.098	0.000	0.454	0.000	1.108	0.000	0.476	0.000	1.117	0.000	0.499	0.000	0.00
0.02	1.081	0.008	0.411	0.022	1.089	0.009	0.432	0.022	1.098	0.009	0.454	0.022	1.108	0.009	0.476	0.022	1.117	0.010	0.498	0.022	0.02
0.04	1.080	0.016	0.410	0.043	1.088	0.017	0.431	0.044	1.097	0.018	0.453	0.044	1.107	0.019	0.475	0.044	1.116	0.020	0.498	0.045	0.04
0.06	1.079	0.025	0.410	0.065	1.087	0.026	0.431	0.065	1.096	0.027	0.452	0.066	1.106	0.029	0.474	0.066	1.115	0.030	0.498	0.067	0.06
0.08	1.078	0.033	0.409	0.086	1.086	0.035	0.430	0.087	1.095	0.036	0.452	0.088	1.105	0.038	0.474	0.089	1.113	0.040	0.497	0.089	0.08
0.10	1.076	0.041	0.409	0.108	1.084	0.043	0.430	0.109	1.093	0.045	0.451	0.110	1.102	0.048	0.473	0.111	1.110	0.050	0.496	0.112	0.10
0.12	1.073	0.049	0.408	0.129	1.082	0.052	0.429	0.130	1.091	0.054	0.450	0.132	1.101	0.057	0.472	0.133	1.108	0.060	0.495	0.134	0.12
0.14	1.070	0.057	0.407	0.151	1.079	0.061	0.428	0.152	1.088	0.063	0.449	0.154	1.098	0.067	0.471	0.155	1.105	0.070	0.494	0.156	0.14
0.16	1.067	0.066	0.405	0.172	1.075	0.069	0.427	0.174	1.085	0.072	0.448	0.175	1.095	0.076	0.470	0.176	1.102	0.079	0.493	0.178	0.16
0.18	1.064	0.074	0.404	0.194	1.072	0.078	0.426	0.196	1.081	0.081	0.446	0.197	1.091	0.085	0.468	0.198	1.098	0.089	0.491	0.200	0.18
0.20	1.059	0.082	0.403	0.215	1.068	0.086	0.424	0.217	1.076	0.090	0.444	0.219	1.086	0.095	0.466	0.220	1.093	0.099	0.489	0.222	0.20
0.22	1.055	0.090	0.401	0.236	1.063	0.094	0.422	0.238	1.072	0.099	0.442	0.241	1.081	0.104	0.464	0.242	1.089	0.109	0.487	0.244	0.22
0.24	1.050	0.098	0.399	0.257	1.058	0.103	0.420	0.259	1.067	0.108	0.440	0.262	1.076	0.113	0.462	0.264	1.083	0.118	0.485	0.266	0.24
0.26	1.044	0.106	0.397	0.278	1.053	0.111	0.418	0.281	1.062	0.117	0.438	0.283	1.071	0.123	0.460	0.284	1.077	0.129	0.482	0.287	0.26
0.28	1.039	0.114	0.395	0.299	1.047	0.120	0.415	0.302	1.056	0.126	0.435	0.304	1.065	0.132	0.457	0.306	1.071	0.138	0.479	0.309	0.28
0.30	1.033	0.121	0.392	0.319	1.040	0.128	0.412	0.322	1.049	0.134	0.432	0.325	1.058	0.141	0.454	0.327	1.065	6.147	0.476	0.330	0.30
0.32	1.026	0.129	0.389	0.340	1.034	0.136	0.409	0.343	1.042	0.143	0.429	0.346	1.051	0.150	0.451	0.348	1.059	0.157	0.473	0.351	0.32
0.34	1.019	0.138	0.386	0.360	1.027	0.144	0.406	0.364	1.036	0.152	0.426	0.367	1.044	0.159	0.448	0.370	1.052	0.166	0.470	0.372	0.34
0.36	1.012	0.145	0.383	0.381	1.020	0.152	0.403	0.384	1.029	0.160	0.423	0.387	1.037	0.168	0.445	0.390	1.044	0.176	0.467	0.393	0.36
0.38	1.004	0.152	0.381	0.401	1.012	0.160	0.401	0.404	1.021	0.168	0.420	0.407	1.029	0.177	0.442	0.410	1.037	0.185	0.464	0.414	0.38
0.40	0.996	0.160	0.378	0.421	1.003	0.168	0.398	0.424	1.012	0.176	0.417	0.427	1.020	0.185	0.439	0.430	1.029	0.194	0.460	0.435	0.40
0.42	0.987	0.168	0.375	0.441	0.995	0.176	0.395	0.444	1.003	0.185	0.414	0.447	1.011	0.194	0.435	0.450	1.019	0.203	0.456	0.455	0.42
0.44	0.978	0.175	0.372	0.460	0.986	0.184	0.392	0.464	0.994	0.193	0.411	0.467	1.002	0.203	0.431	0.470	1.010	0.212	0.452	0.475	0.44
0.46	0.968	0.182	0.368	0.479	0.976	0.192	0.388	0.484	0.985	0.201	0.407	0.487	0.993	0.211	0.427	0.490	1.000	0.221	0.448	0.495	0.46
0.48	0.959	0.190	0.364	0.498	0.966	0.200	0.384	0.503	0.975	0.209	0.403	0.507	0.983	0.220	0.423	0.510	0.991	0.230	0.443	0.515	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.40.$				$x = 0.42.$				$x = 0.44.$				$x = 0.46.$				$x = 0.48.$				y
	a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	
0.50	0.949	0.197	0.360	0.518	0.957	0.207	0.379	0.522	0.965	0.217	0.399	0.526	0.973	0.228	0.418	0.530	0.981	0.239	0.438	0.535	0.50
0.52	0.938	0.204	0.356	0.537	0.946	0.215	0.375	0.541	0.954	0.225	0.395	0.545	0.962	0.236	0.413	0.549	0.970	0.249	0.433	0.554	0.52
0.54	0.927	0.212	0.352	0.555	0.934	0.222	0.371	0.560	0.942	0.233	0.390	0.564	0.950	0.244	0.408	0.568	0.958	0.257	0.428	0.573	0.54
0.56	0.916	0.219	0.348	0.574	0.923	0.230	0.367	0.578	0.930	0.241	0.385	0.582	0.938	0.252	0.403	0.587	0.946	0.265	0.423	0.592	0.56
0.58	0.904	0.225	0.344	0.592	0.911	0.236	0.362	0.596	0.918	0.249	0.380	0.601	0.926	0.260	0.398	0.606	0.934	0.274	0.418	0.611	0.58
0.60	0.892	0.232	0.339	0.610	0.899	0.244	0.357	0.614	0.906	0.256	0.374	0.619	0.914	0.268	0.392	0.624	0.922	0.282	0.412	0.630	0.60
0.62	0.880	0.239	0.334	0.628	0.887	0.250	0.352	0.632	0.893	0.264	0.369	0.637	0.901	0.276	0.387	0.642	0.909	0.290	0.406	0.648	0.62
0.64	0.867	0.245	0.329	0.645	0.874	0.258	0.347	0.650	0.880	0.271	0.364	0.655	0.888	0.284	0.382	0.660	0.896	0.298	0.400	0.666	0.64
0.66	0.854	0.251	0.324	0.662	0.861	0.265	0.342	0.667	0.867	0.278	0.359	0.673	0.874	0.292	0.376	0.678	0.882	0.306	0.394	0.684	0.66
0.68	0.840	0.258	0.319	0.679	0.847	0.272	0.337	0.684	0.854	0.285	0.353	0.690	0.861	0.300	0.370	0.696	0.869	0.314	0.388	0.702	0.68
0.70	0.827	0.265	0.314	0.696	0.833	0.278	0.331	0.701	0.840	0.292	0.347	0.707	0.847	0.307	0.364	0.713	0.855	0.321	0.382	0.719	0.70
0.72	0.813	0.270	0.309	0.712	0.819	0.285	0.325	0.717	0.826	0.299	0.341	0.724	0.833	0.314	0.358	0.730	0.841	0.329	0.376	0.736	0.72
0.74	0.798	0.276	0.304	0.728	0.804	0.292	0.319	0.733	0.811	0.306	0.335	0.740	0.818	0.321	0.352	0.746	0.825	0.337	0.369	0.753	0.74
0.76	0.784	0.283	0.298	0.744	0.790	0.298	0.313	0.749	0.796	0.313	0.329	0.756	0.802	0.328	0.346	0.762	0.809	0.344	0.362	0.769	0.76
0.78	0.769	0.289	0.292	0.760	0.775	0.304	0.307	0.765	0.781	0.319	0.322	0.772	0.787	0.335	0.339	0.778	0.794	0.351	0.355	0.785	0.78
0.80	0.753	0.295	0.286	0.775	0.759	0.310	0.301	0.781	0.766	0.325	0.315	0.788	0.772	0.341	0.332	0.794	0.779	0.358	0.348	0.801	0.80
0.82	0.737	0.300	0.280	0.790	0.743	0.315	0.295	0.796	0.750	0.331	0.309	0.803	0.756	0.348	0.325	0.809	0.763	0.365	0.341	0.816	0.82
0.84	0.721	0.306	0.274	0.805	0.727	0.322	0.289	0.811	0.734	0.337	0.302	0.818	0.740	0.354	0.318	0.824	0.747	0.372	0.334	0.831	0.84
0.86	0.705	0.311	0.268	0.819	0.710	0.328	0.283	0.825	0.717	0.343	0.295	0.832	0.723	0.360	0.311	0.839	0.729	0.378	0.326	0.846	0.86
0.88	0.689	0.316	0.262	0.833	0.694	0.334	0.276	0.839	0.700	0.349	0.288	0.846	0.706	0.366	0.304	0.853	0.712	0.384	0.318	0.861	0.88
0.90	0.672	0.322	0.255	0.847	0.678	0.339	0.269	0.853	0.683	0.355	0.281	0.860	0.689	0.372	0.296	0.867	0.695	0.390	0.310	0.875	0.90
0.92	0.655	0.328	0.249	0.860	0.660	0.343	0.262	0.866	0.666	0.361	0.274	0.873	0.672	0.378	0.289	0.880	0.678	0.396	0.302	0.889	0.92
0.94	0.638	0.332	0.243	0.873	0.643	0.349	0.255	0.879	0.648	0.366	0.267	0.886	0.653	0.384	0.281	0.893	0.659	0.402	0.294	0.902	0.94
0.96	0.621	0.336	0.236	0.885	0.625	0.354	0.248	0.892	0.630	0.371	0.260	0.899	0.635	0.390	0.273	0.906	0.641	0.408	0.286	0.915	0.96
0.98	0.602	0.341	0.229	0.897	0.607	0.359	0.241	0.904	0.612	0.376	0.253	0.912	0.617	0.395	0.265	0.919	0.623	0.414	0.278	0.928	0.98
1.00	0.584	0.346	0.222	0.910	0.588	0.364	0.234	0.916	0.593	0.381	0.245	0.924	0.598	0.400	0.257	0.931	0.604	0.420	0.270	0.940	1.00

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy)\ c + jd$

y	$x = 0.50.$				$x = 0.52.$				$x = 0.54.$				$x = 0.56.$				$x = 0.58.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.128	0.000	0.521	0.000	1.138	0.000	0.544	0.000	1.149	0.000	0.567	0.000	1.161	0.000	0.590	0.000	1.173	0.000	0.613	0.000	0.00
0.02	1.127	0.010	0.521	0.023	1.137	0.011	0.544	0.023	1.149	0.011	0.567	0.023	1.161	0.012	0.590	0.023	1.173	0.012	0.613	0.023	0.02
0.04	1.126	0.021	0.521	0.045	1.136	0.022	0.543	0.045	1.148	0.023	0.566	0.046	1.160	0.024	0.589	0.046	1.172	0.024	0.612	0.047	0.04
0.06	1.125	0.031	0.520	0.068	1.135	0.033	0.542	0.068	1.146	0.034	0.565	0.069	1.159	0.035	0.588	0.070	1.171	0.037	0.611	0.070	0.06
0.08	1.124	0.042	0.519	0.090	1.133	0.043	0.542	0.091	1.144	0.045	0.564	0.092	1.158	0.047	0.587	0.093	1.170	0.049	0.611	0.094	0.08
0.10	1.122	0.052	0.518	0.113	1.131	0.054	0.541	0.114	1.142	0.057	0.563	0.115	1.157	0.059	0.586	0.116	1.169	0.061	0.610	0.117	0.10
0.12	1.119	0.062	0.517	0.136	1.129	0.065	0.540	0.136	1.139	0.068	0.562	0.138	1.155	0.071	0.585	0.139	1.167	0.074	0.608	0.141	0.12
0.14	1.116	0.072	0.516	0.158	1.126	0.076	0.539	0.159	1.136	0.079	0.560	0.161	1.152	0.083	0.584	0.163	1.164	0.086	0.606	0.164	0.14
0.16	1.113	0.082	0.515	0.180	1.122	0.086	0.537	0.181	1.133	0.090	0.558	0.183	1.149	0.094	0.582	0.185	1.160	0.098	0.604	0.187	0.16
0.18	1.109	0.092	0.513	0.202	1.118	0.097	0.535	0.204	1.129	0.102	0.556	0.206	1.145	0.106	0.580	0.208	1.155	0.110	0.602	0.211	0.18
0.20	1.105	0.102	0.511	0.224	1.114	0.108	0.533	0.227	1.125	0.112	0.554	0.229	1.140	0.118	0.578	0.231	1.150	0.122	0.600	0.234	0.20
0.22	1.100	0.113	0.509	0.246	1.109	0.119	0.531	0.249	1.120	0.124	0.552	0.250	1.135	0.129	0.576	0.253	1.145	0.134	0.598	0.256	0.22
0.24	1.095	0.123	0.507	0.268	1.104	0.130	0.529	0.271	1.115	0.135	0.549	0.273	1.129	0.141	0.573	0.276	1.140	0.146	0.595	0.279	0.24
0.26	1.089	0.134	0.504	0.290	1.099	0.141	0.526	0.293	1.110	0.146	0.546	0.295	1.123	0.152	0.570	0.299	1.134	0.158	0.592	0.302	0.26
0.28	1.083	0.144	0.501	0.312	1.093	0.151	0.523	0.315	1.104	0.157	0.543	0.318	1.117	0.163	0.567	0.322	1.127	0.170	0.589	0.325	0.28
0.30	1.077	0.154	0.498	0.333	1.087	0.161	0.520	0.337	1.098	0.168	0.540	0.340	1.110	0.174	0.564	0.344	1.120	0.182	0.585	0.348	0.30
0.32	1.070	0.164	0.495	0.355	1.080	0.172	0.516	0.359	1.091	0.179	0.537	0.362	1.103	0.185	0.560	0.366	1.113	0.194	0.581	0.370	0.32
0.34	1.063	0.174	0.492	0.376	1.072	0.182	0.512	0.380	1.083	0.190	0.534	0.384	1.095	0.196	0.556	0.388	1.106	0.206	0.577	0.392	0.34
0.36	1.055	0.184	0.488	0.397	1.064	0.192	0.508	0.401	1.075	0.200	0.530	0.405	1.087	0.208	0.552	0.410	1.098	0.217	0.573	0.414	0.36
0.38	1.047	0.194	0.484	0.418	1.056	0.202	0.504	0.422	1.067	0.210	0.526	0.426	1.079	0.219	0.548	0.431	1.090	0.228	0.569	0.436	0.38
0.40	1.039	0.203	0.480	0.439	1.048	0.212	0.500	0.443	1.058	0.220	0.521	0.446	1.070	0.230	0.543	0.452	1.081	0.239	0.565	0.457	0.40
0.42	1.030	0.213	0.476	0.460	1.039	0.222	0.496	0.464	1.049	0.231	0.516	0.468	1.061	0.241	0.538	0.474	1.072	0.250	0.560	0.479	0.42
0.44	1.021	0.222	0.472	0.481	1.029	0.232	0.492	0.485	1.040	0.241	0.511	0.490	1.051	0.252	0.533	0.495	1.062	0.261	0.555	0.500	0.44
0.46	1.011	0.232	0.467	0.501	1.019	0.242	0.488	0.505	1.030	0.251	0.506	0.510	1.041	0.263	0.528	0.516	1.052	0.272	0.550	0.521	0.46
0.48	1.001	0.241	0.462	0.521	1.008	0.251	0.483	0.525	1.019	0.261	0.501	0.531	1.031	0.273	0.523	0.537	1.042	0.283	0.545	0.542	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	x = 0.50.				x = 0.52.				x = 0.54.				x = 0.56.				x = 0.58.				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	0.990	0.250	0.457	0.541	0.997	0.260	0.478	0.545	1.008	0.271	0.496	0.550	1.020	0.283	0.518	0.557	1.031	0.294	0.539	0.563	0.50
0.52	0.979	0.259	0.452	0.561	0.986	0.270	0.473	0.565	0.997	0.281	0.491	0.571	1.009	0.293	0.512	0.577	1.019	0.305	0.533	0.583	0.52
0.54	0.967	0.268	0.447	0.580	0.975	0.279	0.467	0.585	0.985	0.291	0.485	0.590	0.997	0.303	0.506	0.597	1.007	0.316	0.526	0.603	0.54
0.56	0.955	0.277	0.442	0.599	0.963	0.289	0.461	0.604	0.973	0.301	0.479	0.610	0.985	0.313	0.500	0.617	0.995	0.327	0.519	0.623	0.56
0.58	0.943	0.286	0.436	0.618	0.951	0.298	0.455	0.623	0.961	0.310	0.473	0.630	0.973	0.323	0.494	0.637	0.983	0.337	0.512	0.643	0.58
0.60	0.931	0.294	0.430	0.637	0.939	0.307	0.448	0.642	0.948	0.319	0.467	0.648	0.960	0.333	0.487	0.655	0.970	0.347	0.505	0.663	0.60
0.62	0.919	0.303	0.424	0.656	0.926	0.316	0.442	0.661	0.935	0.329	0.461	0.667	0.947	0.343	0.480	0.676	0.957	0.357	0.498	0.682	0.62
0.64	0.906	0.311	0.418	0.674	0.912	0.325	0.436	0.679	0.922	0.338	0.455	0.686	0.933	0.353	0.473	0.694	0.943	0.367	0.491	0.701	0.64
0.66	0.892	0.320	0.412	0.692	0.898	0.334	0.430	0.697	0.908	0.347	0.447	0.704	0.919	0.362	0.466	0.713	0.929	0.377	0.484	0.720	0.66
0.68	0.877	0.328	0.406	0.709	0.884	0.343	0.424	0.715	0.894	0.356	0.440	0.722	0.905	0.371	0.459	0.731	0.914	0.386	0.477	0.738	0.68
0.70	0.862	0.336	0.399	0.726	0.870	0.351	0.417	0.733	0.879	0.365	0.433	0.739	0.890	0.380	0.451	0.749	0.899	0.395	0.469	0.756	0.70
0.72	0.847	0.344	0.392	0.743	0.855	0.359	0.410	0.750	0.864	0.374	0.426	0.757	0.875	0.389	0.443	0.766	0.884	0.404	0.461	0.774	0.72
0.74	0.832	0.352	0.385	0.760	0.840	0.367	0.402	0.767	0.849	0.382	0.418	0.774	0.860	0.398	0.435	0.784	0.868	0.413	0.453	0.792	0.74
0.76	0.817	0.360	0.378	0.777	0.825	0.375	0.394	0.784	0.833	0.390	0.410	0.791	0.844	0.406	0.427	0.801	0.852	0.422	0.445	0.809	0.76
0.78	0.802	0.367	0.371	0.793	0.809	0.383	0.386	0.801	0.817	0.398	0.402	0.808	0.828	0.414	0.419	0.818	0.836	0.431	0.436	0.826	0.78
0.80	0.786	0.374	0.363	0.809	0.793	0.390	0.378	0.817	0.800	0.405	0.394	0.824	0.811	0.423	0.411	0.834	0.819	0.440	0.427	0.843	0.80
0.82	0.770	0.381	0.356	0.824	0.776	0.398	0.371	0.833	0.783	0.413	0.386	0.840	0.794	0.431	0.403	0.850	0.802	0.449	0.418	0.859	0.82
0.84	0.753	0.388	0.348	0.839	0.759	0.405	0.363	0.848	0.766	0.420	0.378	0.855	0.777	0.439	0.394	0.865	0.784	0.457	0.409	0.875	0.84
0.86	0.736	0.395	0.340	0.854	0.742	0.412	0.355	0.863	0.749	0.428	0.370	0.871	0.760	0.447	0.385	0.880	0.766	0.465	0.400	0.890	0.86
0.88	0.719	0.402	0.332	0.869	0.725	0.419	0.347	0.877	0.732	0.435	0.361	0.886	0.742	0.454	0.376	0.895	0.748	0.473	0.391	0.905	0.88
0.90	0.701	0.408	0.324	0.883	0.707	0.426	0.339	0.891	0.714	0.442	0.352	0.900	0.723	0.461	0.367	0.910	0.730	0.481	0.381	0.920	0.90
0.92	0.683	0.414	0.316	0.897	0.689	0.433	0.330	0.905	0.696	0.449	0.343	0.915	0.704	0.468	0.357	0.924	0.712	0.488	0.371	0.934	0.92
0.94	0.665	0.420	0.308	0.910	0.671	0.439	0.331	0.918	0.678	0.456	0.334	0.928	0.685	0.475	0.348	0.938	0.693	0.495	0.361	0.948	0.94
0.96	0.647	0.426	0.299	0.923	0.653	0.445	0.312	0.931	0.659	0.462	0.325	0.940	0.666	0.482	0.339	0.952	0.674	0.502	0.351	0.962	0.96
0.98	0.628	0.432	0.290	0.936	0.634	0.451	0.303	0.944	0.640	0.469	0.315	0.954	0.647	0.489	0.329	0.965	0.655	0.509	0.341	0.975	0.98
1.00	0.609	0.438	0.281	0.949	0.615	0.457	0.293	0.957	0.620	0.475	0.305	0.966	0.627	0.496	0.319	0.977	0.634	0.516	0.331	0.988	1.00

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	x = 0.60.				x = 0.62.				x = 0.64.				x = 0.66.				x = 0.68.				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.185	0.000	0.637	0.000	1.198	0.000	0.660	0.000	1.212	0.000	0.685	0.000	1.226	0.000	0.709	0.000	1.240	0.000	0.734	0.000	0.00
0.02	1.185	0.013	0.637	0.024	1.198	0.013	0.660	0.024	1.212	0.014	0.685	0.024	1.226	0.014	0.709	0.025	1.240	0.015	0.734	0.025	0.02
0.04	1.184	0.025	0.637	0.047	1.197	0.026	0.659	0.048	1.211	0.027	0.684	0.049	1.225	0.028	0.708	0.049	1.239	0.029	0.733	0.050	0.04
0.06	1.183	0.038	0.636	0.071	1.195	0.040	0.659	0.072	1.210	0.041	0.683	0.073	1.224	0.042	0.707	0.074	1.238	0.044	0.732	0.074	0.06
0.08	1.181	0.051	0.635	0.095	1.193	0.053	0.658	0.096	1.209	0.055	0.682	0.097	1.222	0.057	0.706	0.098	1.237	0.059	0.731	0.099	0.08
0.10	1.179	0.064	0.633	0.118	1.191	0.066	0.657	0.120	1.207	0.068	0.681	0.121	1.220	0.071	0.705	0.123	1.235	0.073	0.729	0.124	0.10
0.12	1.176	0.076	0.632	0.142	1.188	0.079	0.656	0.144	1.204	0.082	0.679	0.146	1.217	0.085	0.703	0.147	1.232	0.088	0.727	0.149	0.12
0.14	1.172	0.089	0.630	0.166	1.185	0.092	0.654	0.168	1.201	0.095	0.677	0.170	1.213	0.099	0.701	0.172	1.228	0.103	0.725	0.174	0.14
0.16	1.169	0.101	0.628	0.189	1.181	0.105	0.652	0.190	1.198	0.109	0.675	0.193	1.209	0.113	0.699	0.195	1.224	0.116	0.723	0.197	0.16
0.18	1.166	0.114	0.626	0.212	1.177	0.118	0.650	0.214	1.194	0.122	0.673	0.217	1.205	0.127	0.697	0.219	1.220	0.131	0.721	0.222	0.18
0.20	1.162	0.126	0.624	0.235	1.173	0.131	0.648	0.238	1.190	0.136	0.670	0.241	1.200	0.141	0.694	0.244	1.215	0.146	0.719	0.247	0.20
0.22	1.158	0.139	0.622	0.258	1.169	0.144	0.645	0.261	1.185	0.149	0.667	0.264	1.195	0.154	0.691	0.267	1.210	0.161	0.716	0.270	0.22
0.24	1.153	0.151	0.619	0.281	1.164	0.157	0.642	0.285	1.180	0.163	0.664	0.289	1.190	0.169	0.688	0.292	1.204	0.175	0.713	0.295	0.24
0.26	1.147	0.164	0.616	0.304	1.158	0.170	0.639	0.308	1.174	0.176	0.661	0.313	1.184	0.182	0.684	0.315	1.198	0.189	0.709	0.319	0.26
0.28	1.140	0.176	0.612	0.327	1.151	0.183	0.636	0.332	1.167	0.190	0.658	0.336	1.177	0.196	0.680	0.340	1.192	0.203	0.705	0.344	0.28
0.30	1.132	0.188	0.608	0.350	1.143	0.196	0.631	0.354	1.159	0.202	0.654	0.360	1.170	0.210	0.676	0.863	1.185	0.217	0.700	0.367	0.30
0.32	1.125	0.200	0.604	0.373	1.135	0.208	0.627	0.377	1.151	0.215	0.650	0.383	1.163	0.223	0.672	0.386	1.177	0.231	0.695	0.391	0.32
0.34	1.117	0.212	0.600	0.396	1.127	0.220	0.623	0.400	1.143	0.228	0.645	0.405	1.155	0.236	0.667	0.409	1.169	0.245	0.690	0.414	0.34
0.36	1.109	0.224	0.596	0.418	1.119	0.233	0.619	0.422	1.135	0.241	0.640	0.428	1.147	0.250	0.662	0.432	1.160	0.259	0.685	0.437	0.36
0.38	1.101	0.236	0.591	0.440	1.111	0.245	0.614	0.444	1.126	0.254	0.635	0.450	1.138	0.263	0.657	0.455	1.151	0.272	0.680	0.460	0.38
0.40	1.092	0.248	0.586	0.462	1.102	0.257	0.609	0.466	1.117	0.266	0.630	0.472	1.129	0.276	0.652	0.477	1.141	0.285	0.675	0.483	0.40
0.42	1.082	0.260	0.581	0.484	1.093	0.270	0.604	0.488	1.107	0.279	0.625	0.496	1.119	0.289	0.646	0.499	1.131	0.299	0.669	0.506	0.42
0.44	1.072	0.272	0.576	0.505	1.083	0.282	0.698	0.510	1.097	0.292	0.619	0.518	1.109	0.302	0.640	0.521	1.121	0.312	0.663	0.528	0.44
0.46	1.062	0.283	0.571	0.526	1.072	0.293	0.592	0.532	1.087	0.305	0.613	0.539	1.098	0.315	0.634	0.543	1.110	0.325	0.657	0.550	0.46
0.48	1.051	0.294	0.565	0.547	1.061	0.305	0.586	0.553	1.076	0.317	0.607	0.561	1.087	0.327	0.628	0.565	1.099	0.338	0.650	0.572	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.60.$				$x = 0.62.$				$x = 0.64.$				$x = 0.66.$				$x = 0.68.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	1.040	0.305	0.559	0.568	1.050	0.316	0.580	0.574	1.065	0.329	0.601	0.581	1.076	0.339	0.622	0.587	1.088	0.351	0.643	0.594	0.50
0.52	1.028	0.316	0.553	0.589	1.038	0.328	0.573	0.595	1.053	0.341	0.595	0.603	1.064	0.351	0.615	0.608	1.076	0.364	0.636	0.615	0.52
0.54	1.016	0.327	0.546	0.610	1.026	0.340	0.566	0.615	1.040	0.353	0.588	0.624	1.051	0.363	0.608	0.629	1.063	0.377	0.629	0.636	0.54
0.56	1.004	0.338	0.539	0.630	1.014	0.351	0.559	0.635	1.027	0.364	0.581	0.645	1.038	0.375	0.601	0.650	1.050	0.389	0.621	0.657	0.56
0.58	0.991	0.349	0.532	0.650	1.001	0.362	0.552	0.655	1.014	0.375	0.573	0.666	1.024	0.387	0.593	0.671	1.036	0.401	0.613	0.678	0.58
0.60	0.978	0.359	0.525	0.669	0.988	0.373	0.545	0.675	1.000	0.386	0.565	0.686	1.010	0.399	0.585	0.691	1.022	0.413	0.605	0.699	0.60
0.62	0.964	0.370	0.518	0.688	0.974	0.384	0.537	0.695	0.986	0.397	0.557	0.706	0.996	0.411	0.577	0.711	1.008	0.425	0.597	0.719	0.62
0.64	0.950	0.380	0.511	0.707	0.960	0.394	0.529	0.715	0.972	0.408	0.549	0.725	0.982	0.423	0.569	0.731	0.994	0.437	0.588	0.739	0.64
0.66	0.936	0.390	0.504	0.726	0.946	0.405	0.521	0.734	0.959	0.419	0.541	0.745	0.968	0.434	0.560	0.751	0.980	0.449	0.579	0.759	0.66
0.68	0.922	0.400	0.496	0.745	0.931	0.415	0.513	0.753	0.944	0.430	0.533	0.764	0.954	0.445	0.551	0.771	0.965	0.461	0.570	0.779	0.68
0.70	0.907	0.410	0.487	0.764	0.916	0.425	0.505	0.771	0.929	0.440	0.524	0.782	0.939	0.456	0.542	0.790	0.950	0.472	0.560	0.799	0.70
0.72	0.891	0.420	0.479	0.782	0.901	0.435	0.497	0.789	0.912	0.451	0.515	0.800	0.923	0.467	0.533	0.809	0.934	0.483	0.551	0.818	0.72
0.74	0.875	0.430	0.471	0.799	0.885	0.445	0.488	0.808	0.895	0.461	0.506	0.819	0.907	0.478	0.524	0.827	0.917	0.494	0.541	0.836	0.74
0.76	0.859	0.439	0.462	0.816	0.869	0.455	0.479	0.825	0.879	0.471	0.497	0.837	0.890	0.488	0.514	0.845	0.900	0.505	0.531	0.854	0.76
0.78	0.843	0.448	0.453	0.833	0.852	0.464	0.470	0.842	0.863	0.481	0.487	0.854	0.873	0.498	0.504	0.863	0.883	0.516	0.521	0.872	0.78
0.80	0.826	0.457	0.444	0.850	0.835	0.474	0.460	0.858	0.846	0.491	0.477	0.871	0.855	0.508	0.494	0.880	0.865	0.526	0.511	0.890	0.80
0.82	0.809	0.466	0.435	0 866	0.818	0.483	0.451	0.874	0.829	0.500	0.467	0.888	0.837	0.518	0.484	0.897	0.847	0.536	0.500	0.907	0.82
0.84	0.791	0.475	0.426	0.882	0.800	0.491	0.441	0.890	0.812	0.509	0.457	0.904	0.819	0.527	0.474	0.913	0.829	0.546	0.489	0.924	0.84
0.86	0.773	0.483	0.416	0.898	0.782	0.501	0.431	0.906	0.794	0.518	0.447	0.921	0.801	0.536	0.463	0.929	0.810	0.556	0.478	0.940	0.86
0.88	0.755	0.491	0.406	0.914	0.764	0.510	0.421	0.922	0.775	0.527	0.437	0.936	0.782	0.545	0.452	0.945	0.791	0.565	0.467	0.956	0.88
0.90	0.737	0.499	0.396	0.929	0.745	0.518	0.411	0.938	0.755	0.536	0.426	0.951	0.763	0.554	0.441	0.960	0.772	0.574	0.456	0.972	0.90
0.92	0.718	0.507	0.386	0.944	0.726	0.525	0.401	0.953	0.735	0.545	0.415	0.966	0.743	0.563	0.430	0.975	0.752	0.583	0.444	0.987	0.92
0.94	0.699	0.515	0.376	0.958	0.707	0.533	0.390	0.967	0.715	0.553	0.404	0.981	0.723	0.572	0.418	0.989	0.732	0.592	0.432	1.001	0.94
0.96	0.680	0.522	0.366	0.972	0.687	0.541	0.379	0.981	0.695	0.561	0.393	0.995	0.703	0.580	0.406	1.003	0.712	0.600	0.420	1.015	0.96
0.98	0.661	0.529	0.355	0.985	0.667	0.549	0.368	0.994	0.675	0.568	0.382	1.008	0.683	0.588	0.394	1.017	0.691	0.608	0.408	1.029	0.98
1.00	0.641	0.536	0.344	0.998	0.646	0:556	0.357	1.008	0.655	0.575	0.370	1.020	0.662	0.596	0.382	1.031	0.670	0.616	0.396	1.043	1.00

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.70.$				$x = 0.72.$				$x = 0.74.$				$x = 0.76.$				$x = 0.78.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.255	0.000	0.759	0.000	1.271	0.000	0.784	0.000	1.287	0.000	0.809	0.000	1.303	0.000	0.835	0.000	1.320	0.000	0.861	0.000	0.00
0.02	1.255	0.015	0.759	0.025	1.271	0.016	0.784	0.025	1.286	0.016	0.809	0.026	1.303	0.017	0.835	0.026	1.319	0.017	0.861	0.026	0.02
0.04	1.254	0.030	0.758	0.050	1.270	0.031	0.783	0.051	1.285	0.032	0.808	0.051	1.302	0.033	0.834	0.052	1.318	0.034	0.860	0.053	0.04
0.06	1.253	0.045	0.757	0.075	1.269	0.047	0.782	0.076	1.284	0.049	0.806	0.077	1.301	0.050	0.833	0.078	1.317	0.052	0.859	0.079	0.06
0.08	1.251	0.061	0.756	0.100	1.267	0.063	0.781	0.102	1.282	0.065	0.806	0.103	1.300	0.067	0.832	0.104	1.316	0.069	0.858	0.106	0.08
0.10	1.249	0.076	0.755	0.125	1.265	0.078	0.780	0.127	1.280	0.081	0.805	0.129	1.298	0.083	0.831	0.130	1.314	0.086	0.857	0.132	0.10
0.12	1.246	0.091	0.753	0.150	1.262	0.094	0.778	0.153	1.277	0.097	0.803	0.154	1.295	0.100	0.829	0.156	1.311	0.103	0.855	0.158	0.12
0.14	1.242	0.106	0.751	0.176	1.259	0.110	0.776	0.178	1.274	0.113	0.801	0.180	1.292	0.117	0.827	0.183	1.307	0.121	0.853	0.185	0.14
0.16	1.239	0.121	0.749	0.200	1.255	0.125	0.773	0.202	1.270	0.129	0.798	0.204	1.288	0.133	0.825	0.207	1.303	0.137	0.851	0.210	0.16
0.18	1.235	0.136	0.746	0.224	1.251	0.140	0.770	0.228	1.265	0.145	0.795	0.230	1.284	0.149	0.822	0.234	1.299	0.154	0.848	0.236	0.18
0.20	1.230	0.151	0.743	0.249	1.247	0.156	0.767	0.253	1.260	0.161	0.792	0.256	1.279	0.166	0.819	0.260	1.294	0.171	0.845	0.263	0.20
0.22	1.225	0.165	0.740	0.274	1.242	0.172	0.764	0.278	1.255	0.176	0.788	0.280	1.273	0.182	0.815	0.284	1.288	0.188	0.841	0.288	0.22
0.24	1.219	0.180	0.737	0.298	1.236	0.187	0.760	0.303	1.249	0.192	0.784	0.306	1.267	0.199	0.811	0.310	1.282	0.205	0.837	0.314	0.24
0.26	1.212	0.195	0.733	0.323	1.229	0.202	0.756	0.328	1.243	0.208	0.780	0.331	1.260	0.215	0.807	0.335	1.275	0.221	0.833	0.339	0.26
0.28	1.206	0.210	0.729	0.348	1.222	0.217	0.752	0.353	1.236	0.224	0.776	0.356	1.253	0.231	0.802	0.361	1.268	0.239	0.828	0.366	0.28
0.30	1.199	0.224	0.725	0.371	1.214	0.232	0.748	0.377	1.229	0.239	0.772	0.381	1.245	0.247	0.797	0.386	1.261	0.255	0.823	0.391	0.30
0.32	1.191	0.239	0.720	0.395	1.206	0.247	0.743	0.401	1.221	0.255	0.767	0.405	1.237	0.263	0.792	0.411	1.252	0.271	0.818	0.416	0.32
0.34	1.183	0.254	0.715	0.418	1.198	0.262	0.738	0.425	1.212	0.270	0.762	0.430	1.228	0.279	0.787	0.435	1.243	0.287	0.812	0.440	0.34
0.36	1.174	0.268	0.710	0.442	1.189	0.277	0.733	0.449	1.203	0.285	0.757	0.454	1.219	0.295	0.782	0.460	1.234	0.303	0.806	0.466	0.36
0.38	1.165	0.281	0.704	0.465	1.180	0.291	0.727	0.472	1.193	0.300	0.752	0.477	1.210	0.310	0.776	0.484	1.225	0.319	0.800	0.490	0.38
0.40	1.156	0.295	0.699	0.489	1.171	0.305	0.721	0.495	1.183	0.315	0.746	0.500	1.200	0.325	0.770	0.508	1.215	0.335	0.794	0.514	0.40
0.42	1.146	0.309	0.693	0.512	1.162	0.319	0.715	0.518	1.173	0.330	0.739	0.524	1.190	0.340	0.763	0.533	1.204	0.351	0.787	0.538	0.42
0.44	1.136	0.323	0.687	0.535	1.152	0.333	0.709	0.541	1.162	0.345	0.732	0.547	1.180	0.355	0.756	0.556	1.193	0.367	0.780	0.562	0.44
0.46	1.125	0.336	0.680	0.558	1.141	0.347	0.702	0.564	1.151	0.359	0.725	0.570	1.169	0.370	0.749	0.580	1.182	0.383	0.772	0.586	0.46
0.48	1.114	0.350	0.673	0.580	1.129	0.361	0.695	0.587	1.140	0.373	0.718	0.593	1.158	0.385	0.741	0.603	1.171	0.398	0.764	0.609	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.70.$				$x = 0.72.$				$x = 0.74.$				$x = 0.76.$				$x = 0.78.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	1.102	0.364	0.666	0.602	1.117	0.375	0.688	0.609	1.129	0.387	0.710	0.616	1.146	0.399	0.733	0.625	1.159	0.413	0.756	0.632	0.50
0.52	1.090	0.377	0.658	0.624	1.104	0.389	0.680	0.631	1.116	0.401	0.702	0.638	1.133	0.414	0.725	0.649	1.146	0.428	0.747	0.655	0.52
0.54	1.077	0.389	0.650	0.646	1.091	0.402	0.672	0.653	1.103	0.415	0.694	0.660	1.120	0.428	0.716	0.670	1.132	0.443	0.738	0.678	0.54
0.56	1.064	0.402	0.642	0.667	1.078	0.415	0.664	0.675	1.090	0.429	0.685	0.682	1.106	0.443	0.707	0.692	1.118	0.457	0.729	0.700	0.56
0.58	1.050	0.415	0.634	0.688	1.064	0.428	0.655	0.697	1.076	0.443	0.676	0.704	1.092	0.457	0.698	0.714	1.104	0.471	0.720	0.722	0.58
0.60	1.036	0.427	0.626	0.709	1.050	0.441	0.646	0.718	1.061	0.456	0.667	0.725	1.077	0.471	0.689	0.735	1.090	0.485	0.711	0.744	0.60
0.62	1.019	0.440	0.617	0.729	1.035	0.454	0.637	0.739	1.046	0.469	0.658	0.746	1.062	0.485	0.680	0.758	1.075	0.499	0.701	0.766	0.62
0.64	1.002	0.453	0.608	0.749	1.020	0.467	0.628	0.759	1.031	0.482	0.649	0.767	1.047	0.498	0.670	0.779	1.059	0.513	0.691	0.787	0.64
0.66	0.985	0.465	0.599	0.769	1.005	0.480	0.619	0.779	1.016	0.495	0.639	0.788	1.031	0.511	0.660	0.800	1.043	0.527	0.681	0.808	0.66
0.68	0.978	0.477	0.590	0.789	0.990	0.492	0.609	0.799	1.001	0.509	0.629	0.808	1.015	0.524	0.650	0.820	1.027	0.541	0.670	0.829	0.68
0.70	0.960	0.489	0.580	0.809	0.974	0.504	0.599	0.819	0.985	0.521	0.619	0.828	0.999	0.537	0.639	0.840	1.011	0.555	0.659	0.850	0.70
0.72	0.943	0.501	0.570	0.828	0.957	0.516	0.589	0.838	0.968	0.533	0.609	0.848	0.982	0.550	0.628	0.860	0.994	0.568	0.648	0.870	0.72
0.74	0.926	0.512	0.560	0.846	0.940	0.528	0.579	0.857	0.951	0.545	0.598	0.867	0.965	0.562	0.617	0.880	0.976	0.581	0.637	0.890	0.74
0.76	0.909	0.523	0.550	0.864	0.923	0.540	0.568	0.876	0.933	0.557	0.587	0.886	0.947	0.574	0.606	0.899	0.958	0.594	0.625	0.909	0.76
0.78	0.892	0.534	0.539	0.882	0.905	0.551	0.557	0.894	0.915	0.569	0.576	0.905	0.929	0.586	0.594	0.918	0.940	0.606	0.613	0.928	0.78
0.80	0.874	0.544	0.529	0.900	0.887	0.562	0.546	0.912	0.897	0.580	0.564	0.923	0.910	0.598	0.582	0.936	0.921	0.618	0.601	0.947	0.80
0.82	0.856	0.554	0.517	0.917	0.869	0.573	0.535	0.930	0.878	0.591	0.552	0.941	0.891	0.610	0.570	0.954	0.902	0.630	0.589	0.965	0.82
0.84	0.837	0.564	0.506	0.934	0.850	0.583	0.523	0.947	0.859	0.602	0.540	0.958	0.872	0.621	0.558	0.971	0.882	0.641	0.576	0.983	0.84
0.86	0.818	0.574	0.494	0.951	0.831	0.593	0.511	0.964	0.840	0.613	0.528	0.975	0.852	0.632	0.546	0.988	0.862	0.652	0.563	1.000	0.86
0.88	0.799	0.584	0.483	0.967	0.812	0.603	0.499	0.980	0.820	0.623	0.516	0.991	0.832	0.643	0.533	1.004	0.842	0.663	0.550	1.017	0.88
0.90	0.780	0.594	0.472	0.983	0.792	0.613	0.487	0.996	0.800	0.633	0.503	1.007	0.812	0.654	0.520	1.020	0.821	0.674	0.536	1.034	0.90
0.92	0.760	0.603	0.460	0.998	0.771	0.623	0.475	1.013	0.780	0.643	0.490	1.023	0.791	0.664	0.507	1.036	0.800	0.685	0.522	1.050	0.92
0.94	0.740	0.612	0.447	1.013	0.750	0.632	0.462	1.028	0.759	0.653	0.477	1.038	0.770	0.674	0.493	1.051	0.779	0.695	0.508	1.066	0.94
0.96	0.720	0.621	0.435	1.028	0.729	0.641	0.449	1.042	0.738	0.663	0.464	1.053	0.749	0.684	0.479	1.066	0.758	0.705	0.494	1.081	0.96
0.98	0.699	0.630	0.422	1.042	0.708	0.650	0.436	1.057	0.717	0.672	0.450	1.068	0.727	0.693	0.465	1.081	0.736	0.715	0.480	1.096	0.98
1.00	0.678	0.638	0.410	1.056	0.687	0.659	0.423	1.070	0.695	0.681	0.436	1.082	0.705	0.702	0.451	1.096	0.714	0.725	0.465	1.110	1.00

TABLE XX—(*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.80.$				$x = 0.82.$				$x = 0.84.$				$x = 0.86.$				$x = 0.88.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.337	0.000	0.888	0.000	1.355	0.000	0.915	0.000	1.374	0.000	0.942	0.000	1.393	0.000	0.970	0.000	1.413	0.000	0.998	0.000	0.00
0.02	1.337	0.018	0.888	0.027	1.355	0.018	0.915	0.027	1.374	0.019	0.942	0.028	1.393	0.019	0.970	0.028	1.413	0.020	0.998	0.028	0.02
0.04	1.336	0.035	0.887	0.053	1.354	0.037	0.914	0.054	1.373	0.038	0.941	0.055	1.392	0.039	0.969	0.056	1.412	0.040	0.997	0.057	0.04
0.06	1.335	0.053	0.886	0.080	1.353	0.055	0.913	0.081	1.372	0.056	0.940	0.083	1.391	0.058	0.968	0.084	1.411	0.060	0.996	0.085	0.06
0.08	1.333	0.071	0.885	0.107	1.351	0.073	0.912	0.108	1.371	0.075	0.939	0.110	1.390	0.078	0.967	0.112	1.410	0.080	0.995	0.113	0.08
0.10	1.331	0.089	0.884	0.134	1.349	0.091	0.910	0.136	1.369	0.094	0.937	0.137	1.388	0.097	0.965	0.139	1.408	0.100	0.993	0.141	0.10
0.12	1.328	0.106	0.882	0.160	1.346	0.110	0.908	0.163	1.366	0.113	0.935	0.165	1.385	0.116	0.963	0.167	1.405	0.120	0.991	0.170	0.12
0.14	1.324	0.124	0.880	0.187	1.342	0.128	0.906	0.190	1.362	0.132	0.933	0.193	1.381	0.136	0.961	0.195	1.401	0.140	0.988	0.198	0.14
0.16	1.320	0.141	0.877	0.212	1.338	0.145	0.903	0.216	1.358	0.150	0.930	0.219	1.377	0.154	0.958	0.222	1.397	0.159	0.985	0.225	0.16
0.18	1.316	0.159	0.874	0.239	1.334	0.164	0.900	0.243	1.353	0.169	0.927	0.246	1.373	0.173	0.954	0.250	1.392	0.179	0.982	0.253	0.18
0.20	1.311	0.176	0.870	0.266	1.329	0.182	0.896	0.270	1.348	0.187	0.923	0.274	1.368	0.193	0.950	0.278	1.387	0.199	0.978	0.282	0.20
0.22	1.306	0.193	0.867	0.292	1.323	0.199	0.893	0.296	1.342	0.205	0.920	0.300	1.362	0.211	0.946	0.304	1.380	0.218	0.974	0.309	0.22
0.24	1.300	0.211	0.863	0.318	1.317	0.218	0.889	0.323	1.336	0.224	0.916	0.328	1.355	0.231	0.942	0.332	1.373	0.238	0.969	0.337	0.24
0.26	1.293	0.228	0.859	0.344	1.310	0.235	0.885	0.349	1.329	0.242	0.911	0.354	1.348	0.249	0.937	0.359	1.366	0.257	0.964	0.364	0.26
0.28	1.286	0.246	0.854	0.370	1.302	0.253	0.880	0.376	1.321	0.261	0.906	0.381	1.340	0.269	0.932	0.386	1.358	0.276	0.959	0.392	0.28
0.30	1.278	0.262	0.848	0.395	1.294	0.271	0.874	0.401	1.312	0.279	0.900	0.407	1.331	0.287	0.926	0.413	1.350	0.295	0.953	0.419	0.30
0.32	1.270	0.279	0.843	0.421	1.286	0.288	0.869	0.427	1.303	0.297	0.894	0.434	1.322	0.305	0.920	0.440	1.341	0.314	0.947	0.446	0.32
0.34	1.261	0.296	0.837	0.446	1.277	0.306	0.863	0.453	1.295	0.314	0.888	0.460	1.313	0.324	0.914	0.466	1.331	0.333	0.941	0.473	0.34
0.36	1.252	0.313	0.831	0.472	1.268	0.323	0.857	0.479	1.286	0.333	0.881	0.486	1.303	0.342	0.908	0.483	1.321	0.352	0.934	0.500	0.36
0.38	1.242	0.330	0.825	0.496	1.258	0.339	0.850	0.504	1.276	0.349	0.874	0.510	1.293	0.360	0.901	0.518	1.311	0.370	0.927	0.525	0.38
0.40	1.232	0.346	0.818	0.521	1.248	0.356	0.843	0.528	1.265	0.366	0.867	0.535	1.282	0.377	0.894	0.543	1.301	0.388	0.919	0.550	0.40
0.42	1.221	0.362	0.811	0.546	1.237	0.373	0.836	0.554	1.254	0.384	0.860	0.562	1.272	0.395	0.886	0.570	1.290	0.407	0.911	0.577	0.42
0.44	1.210	0.378	0.803	0.570	1.226	0.390	0.828	0.578	1.243	0.401	0.852	0.586	1.261	0.413	0.878	0.595	1.279	0.425	0.903	0.603	0.44
0.46	1.198	0.394	0.795	0.594	1.214	0.406	0.820	0.602	1.232	0.418	0.844	0.610	1.249	0.430	0.869	0.619	1.267	0.443	0.894	0.628	0.46
0.48	1.186	0.410	0.787	0.618	1.202	0.423	0.812	0.627	1.220	0.435	0.836	0.636	1.237	0.447	0.860	0.645	1.255	0.461	0.885	0.653	0.48

TABLE XX — (*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.80.$				$x = 0.82.$				$x = 0.84.$				$x = 0.86.$				$x = 0.88.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	1.174	0.426	0.779	0.641	1.189	0.439	0.803	0.650	1.207	0.451	0.827	0.659	1.224	0.464	0.851	0.668	1.242	0.478	0.876	0.678	0.50
0.52	1.161	0.441	0.770	0.664	1.175	0.455	0.794	0.674	1.193	0.467	0.818	0.684	1.210	0.481	0.841	0.692	1.228	0.496	0.866	0.703	0.52
0.54	1.147	0.456	0.761	0.687	1.161	0.471	0.785	0.697	1.179	0.483	0.808	0.707	1.196	0.498	0.831	0.716	1.214	0.513	0.855	0.727	0.54
0.56	1.133	0.471	0.752	0.710	1.147	0.486	0.775	0.721	1.164	0.499	0.798	0.731	1.181	0.514	0.821	0.740	1.199	0.530	0.845	0.751	0.56
0.58	1.119	0.486	0.743	0.733	1.133	0.501	0.765	0.744	1.149	0.515	0.788	0.754	1.166	0.530	0.811	0.763	1.184	0.547	0.834	0.775	0.58
0.60	1.104	0.501	0.733	0.755	1.118	0.516	0.755	0.765	1.134	0.531	0.777	0.775	1.150	0.546	0.800	0.786	1.168	0.563	0.823	0.799	0.60
0.62	1.089	0.516	0.723	0.777	1.103	0.531	0.745	0.789	1.118	0.546	0.766	0.797	1.135	0.562	0.789	0.809	1.151	0.580	0.812	0.822	0.62
0.64	1.073	0.531	0.712	0.799	1.088	0.546	0.735	0.810	1.102	0.561	0.755	0.819	1.119	0.578	0.788	0.832	1.134	0.596	0.800	0.845	0.64
0.66	1.057	0.545	0.701	0.820	1.072	0.561	0.724	0.832	1.085	0.576	0.744	0.841	1.102	0.594	0.766	0.854	1.117	0.612	0.788	0.867	0.66
0.68	1.040	0.559	0.690	0.841	1.055	0.575	0.712	0.853	1.068	0.591	0.732	0.863	1.085	0.609	0.754	0.876	1.099	0.628	0.775	0.889	0.68
0.70	1.023	0.572	0.679	0.862	1.037	0.589	0.700	0.874	1.051	0.606	0.720	0.885	1.067	0.629	0.742	0.898	1.081	0.643	0.763	0.911	0.70
0.72	1.006	0.585	0.667	0.882	1.020	0.603	0.688	0.894	1.033	0.621	0.708	0.906	1.049	0.639	0.730	0.919	1.063	0.658	0.750	0.932	0.72
0.74	0.988	0.598	0.655	0.902	1.003	0.617	0.676	0.914	1.015	0.635	0.696	0.927	1.031	0.654	0.717	0.940	1.045	0.673	0.736	0.953	0.74
0.76	0.970	0.611	0.643	0.921	0.985	0.630	0.664	0.934	0.997	0.649	0.683	0.947	1.012	0.668	0.704	0.961	1.026	0.688	0.724	0.973	0.76
0.78	0.951	0.624	0.631	0.940	0.966	0.643	0.651	0.954	0.978	0.663	0.670	0.967	0.993	0.682	0.690	0.981	1.007	0.702	0.710	0.993	0.78
0.80	0.932	0.637	0.619	0.959	0.946	0.656	0.638	0.973	0.959	0.676	0.657	0.987	0.973	0.696	0.676	1.001	0.987	0.716	0.696	1.013	0.80
0.82	0.913	0.649	0.606	0.978	0.926	0.669	0.625	0.991	0.939	0.689	0.644	1.006	0.953	0.709	0.662	1.020	0.966	0.730	0.681	1.032	0.82
0.84	0.893	0.661	0.593	0.996	0.906	0.681	0.611	1.009	0.918	0.702	0.630	1.024	0.933	0.722	0.648	1.039	0.945	0.743	0.666	1.051	0.84
0.86	0.873	0.673	0.580	1.014	0.885	0.693	0.597	1.027	0.897	0.714	0.616	1.042	0.912	0.735	0.633	1.057	0.924	0.757	0.651	1.070	0.86
0.88	0.852	0.685	0.566	1.031	0.864	0.705	0.583	1.044	0.876	0.726	0.601	1.060	0.890	0.747	0.618	1.075	0.903	0.769	0.636	1.088	0.88
0.90	0.831	0.696	0.552	1.048	0.843	0.717	0.569	1.061	0.855	0.738	0.586	1.077	0.868	0.759	0.603	1.092	0.881	0.781	0.621	1.106	0.90
0.92	0.810	0.707	0.538	1.064	0.821	0.728	0.554	1.078	0.833	0.749	0.571	1.094	0.846	0.771	0.588	1.109	0.859	0.794	0.605	1.124	0.92
0.94	0.789	0.717	0.524	1.080	0.799	0.739	0.539	1.094	0.811	0.760	0.556	1.110	0.824	0.783	0.572	1.125	0.836	0.806	0.589	1.141	0.94
0.96	0.767	0.727	0.510	1.095	0.777	0.750	0.524	1.110	0.789	0.771	0.541	1.125	0.801	0.794	0.556	1.141	0.813	0.817	0.573	1.158	0.96
0.98	0.745	0.737	0.495	1.110	0.755	0.760	0.509	1.125	0.766	0.782	0.525	1.141	0.778	0.805	0.540	1.157	0.789	0.828	0.556	1.174	0.98
1.00	0.723	0.747	0.480	1.125	0.733	0.770	0.494	1.140	0.743	0.792	0.509	1.156	0.754	0.816	0.524	1.173	0.765	0.839	0.539	1.190	1.00

TABLE XX—(*Continued*)

VALUES OF COSH $(x + jy) = a + jb$ AND SINH $(x + jy) = c + jd$

y	$x = 0.90.$				$x = 0.92.$				$x = 0.94.$				$x = 0.96.$				$x = 0.98.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.00	1.433	0.000	1.026	0.000	1.454	0.000	1.055	0.000	1.475	0.000	1.085	0.000	1.497	0.000	1.114	0.000	1.520	0.000	1.145	0.000	0.00
0.02	1.433	0.021	1.026	0.029	1.454	0.021	1.055	0.029	1.475	0.022	1.085	0.030	1.497	0.022	1.114	0.030	1.519	0.023	1.144	0.030	0.02
0.04	1.432	0.041	1.025	0.057	1.453	0.042	1.054	0.058	1.474	0.043	1.084	0.059	1.496	0.045	1.113	0.060	1.518	0.046	1.144	0.061	0.04
0.06	1.431	0.062	1.024	0.086	1.452	0.063	1.053	0.087	1.473	0.065	1.083	0.089	1.495	0.067	1.112	0.090	1.517	0.069	1.143	0.091	0.06
0.08	1.429	0.082	1.023	0.115	1.450	0.084	1.052	0.116	1.471	0,087	1.081	0.118	1.493	0.089	1.111	0.120	1.515	0.092	1.142	0.122	0.08
0.10	1.426	0.102	1.021	0.143	1.447	0.105	1.050	0.145	1.469	0.108	1.079	0.147	1.490	0.111	1.109	0.150	1.512	0.114	1.139	0.152	0.10
0.12	1.422	0.123	1.019	0.172	1.443	0.127	1.048	0.175	1.466	0.130	1.077	0.177	1.486	0.134	1.107	0.180	1.508	0.137	1.137	0.182	0.12
0.14	1.418	0.144	1.017	0.201	1.439	0.148	1.045	0.204	1.462	0.152	1.075	0.206	1.482	0.156	1.105	0.210	1.504	0.160	1.134	0.213	0.14
0.16	1.414	0.163	1.014	0.228	1.435	0.168	1.042	0.232	1.457	0.172	1.072	0.234	1.477	0.177	1.102	0.238	1.500	0.182	1.131	0.242	0.16
0.18	1.409	0.184	1.010	0.255	1.430	0.189	1.038	0.261	1.452	0.194	1.068	0.264	1.472	0.200	1.098	0.268	1.495	0.205	1.127	0.272	0.18
0.20	1.404	0.204	1.006	0.285	1.425	0.210	1.034	0.290	1.446	0.216	1.063	0.294	1.466	0.222	1.092	0.298	1.489	0.228	1.122	0.303	0.20
0.22	1.398	0.224	1.002	0.313	1.419	0.230	1.030	0.317	1.439	0.236	1.058	0.322	1.459	0.243	1.087	0.326	1.482	0.250	1.117	0.332	0.22
0.24	1.392	0.244	0.997	0.342	1.413	0.251	1.025	0.346	1.432	0.258	1.053	0.351	1.452	0.266	1.082	0.356	1.475	0.273	1.112	0.362	0.24
0.26	1.385	0.264	0.992	0.369	1.406	0.271	1.020	0.374	1.425	0.279	1.048	0.380	1.445	0.287	1.076	0.385	1.467	0.294	1.106	0.391	0.26
0.28	1.377	0.284	0.987	0.398	1.398	0.292	1.014	0.403	1.418	0.300	1.042	0.409	1.437	0.309	1.070	0.415	1.459	0.317	1.100	0.421	0.28
0.30	1.369	0.303	0.981	0.424	1.390	0.312	1.008	0.431	1.410	0.321	1.036	0.436	1.429	0.330	1.064	0.443	1.451	0.339	1.093	0.450	0.30
0.32	1.360	0.322	0.974	0.451	1.381	0.333	1.002	0.458	1.402	0.345	1.029	0.465	1.420	0.351	1.057	0.472	1.442	0.361	1.086	0.479	0.32
0.34	1.351	0.342	0.967	0.478	1.372	0.352	0.995	0.486	1.393	0.362	1.022	0.494	1.411	0.372	1.050	0.500	1.432	0.383	1.079	0.508	0.34
0.36	1.341	0.362	0.960	0.505	1.362	0.373	0.988	0.514	1.382	0.383	1.015	0.521	1.401	0.393	1.042	0.529	1.422	0.405	1.071	0.536	0.36
0.38	1.331	0.381	0.953	0.532	1.352	0.391	0.980	0.540	1.371	0.402	1.007	0.548	1.390	0.414	1.034	0.556	1.411	0.425	1.063	0.564	0.38
0.40	1.320	0.400	0.945	0.558	1.341	0.410	0.972	0.566	1.369	0.422	0.999	0.574	1.378	0.434	1.026	0.583	1.399	0.446	1.054	0.591	0.40
0.42	1.309	0.419	0.937	0.584	1.329	0.431	0.964	0.592	1.347	0.442	0.990	0.601	1.366	0.455	1.017	0.610	1.386	0.467	1.045	0.620	0.42
0.44	1.297	0.438	0.929	0.610	1.317	0.450	0.955	0.618	1.335	0.462	0.981	0.628	1.353	0.475	1.008	0.637	1.373	0.488	1.036	0.648	0.44
0.46	1.285	0.456	0.920	0.636	1.304	0.469	0.946	0.644	1.322	0.481	0.972	0.654	1.340	0.495	0.999	0.664	1.360	0.509	1.026	0.675	0.46
0.48	1.272	0.474	0.911	0.662	1.290	0.487	0.936	0.670	1.308	0.500	0.962	0.680	1.327	0.515	0.989	0.691	1.347	0.529	1.016	0.703	0.48

TABLE XX—(*Concluded*)

VALUES OF COSH $(x+jy)=a+jb$ AND SINH $(x+jy)=c+jd$

y	$x = 0.90.$				$x = 0.92.$				$x = 0.94.$				$x = 0.96.$				$x = 0.98.$				y
	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
0.50	1.258	0.492	0.901	0.687	1.276	0.505	0.927	0.696	1.294	0.519	0.952	0.706	1.313	0.534	0.979	0.717	1.333	0.549	1.005	0.728	0.50
0.52	1.244	0.510	0.891	0.712	1.262	0.523	0.916	0.721	1.279	0.538	0.941	0.732	1.298	0.553	0.968	0.743	1.318	0.569	0.994	0.755	0.52
0.54	1.229	0.528	0.881	0.737	1.247	0.541	0.905	0.746	1.260	0.557	0.930	0.758	1.282	0.572	0.956	0.769	1.303	0.589	0.982	0.781	0.54
0.56	1.214	0.546	0.870	0.761	1.232	0.559	0.894	0.771	1.249	0.576	0.919	0.783	1.266	0.591	0.944	0.795	1.287	0.608	0.970	0.807	0.56
0.58	1.199	0.563	0.859	0.785	1.216	0.577	0.883	0.796	1.233	0.594	0.907	0.808	1.250	0.610	0.932	0.820	1.270	0.627	0.958	0.833	0.58
0.60	1.183	0.580	0.847	0.809	1.200	0.595	0.871	0.821	1.217	0.612	0.895	0.833	1.233	0.629	0.920	0.845	1.253	0.646	0.945	0.858	0.60
0.62	1.166	0.597	0.835	0.833	1.184	0.613	0.859	0.845	1.200	0.630	0.883	0.867	1.216	0.647	0.908	0.869	1.235	0.665	0.932	0.883	0.62
0.64	1.149	0.614	0.823	0.856	1.167	0.630	0.847	0.868	1.182	0.648	0.870	0.891	1.199	0.665	0.895	0.893	1.217	0.684	0.919	0.908	0.64
0.66	1.132	0.630	0.811	0.879	1.149	0.647	0.834	0.891	1.164	0.665	0.857	0.914	1.181	0.683	0.882	0.917	1.199	0.702	0.905	0.932	0.66
0.68	1.114	0.646	0.798	0.901	1.131	0.664	0.821	0.914	1.146	0.682	0.844	0.937	1.162	0.701	0.868	0.941	1.180	0.720	0.891	0.956	0.68
0.70	1.096	0.661	0.785	0.923	1.112	0.680	0.808	0.937	1.128	0.698	0.830	0.950	1.143	0.718	0.854	0.964	1.161	0.738	0.876	0.979	0.70
0.72	1.078	0.676	0.772	0.945	1.093	0.696	0.794	0.959	1.109	0.714	0.816	0.972	1.124	0.735	0.839	0.987	1.142	0.755	0.861	1.002	0.72
0.74	1.059	0.691	0.758	0.966	1.073	0.712	0.780	0.981	1.090	0.730	0.802	0.994	1.105	0.752	0.824	1.009	1.122	0.772	0.846	1.025	0.74
0.76	1.039	0.706	0.744	0.987	1.053	0.727	0.766	1.002	1.070	0.746	0.787	1.016	1.085	0.768	0.809	1.031	1.102	0.789	0.830	1.047	0.76
0.78	1.019	0.721	0.730	1.008	1.033	0.742	0.751	1.023	1.050	0.762	0.772	1.037	1.065	0.784	0.793	1.052	1.081	0.805	0.814	1.069	0.78
0.80	0.998	0.736	0.715	1.028	1.013	0.757	0.736	1.044	1.029	0.778	0.756	1.058	1.044	0.800	0.777	1.073	1.060	0.821	0.798	1.090	0.80
0.82	0.977	0.750	0.700	1.048	0.992	0.772	0.721	1.064	1.008	0.793	0.740	1.078	1.022	0.815	0.761	1.094	1.038	0.837	0.781	1.111	0.82
0.84	0.956	0.764	0.685	1.067	0.971	0.786	0.705	1.083	0.986	0.808	0.724	1.098	1.000	0.830	0.745	1.114	1.016	0.852	0.764	1.131	0.84
0.86	0.935	0.778	0.670	1.086	0.950	0.800	0.689	1.102	0.964	0.822	0.708	1.118	0.978	0.845	0.728	1.134	0.993	0.867	0.747	1.151	0.86
0.88	0.913	0.791	0.654	1.105	0.928	0.813	0.673	1.121	0.941	0.836	0.692	1.137	0.955	0.859	0.711	1.153	0.970	0.882	0.730	1.171	0.88
0.90	0.891	0.804	0.638	1.123	0.906	0.826	0.656	1.139	0.918	0.849	0.675	1.156	0.932	0.873	0.694	1.172	0.946	0.897	0.712	1.190	0.90
0.92	0.868	0.817	0.622	1.140	0.883	0.839	0.639	1.157	0.895	0.862	0.658	1.174	0.908	0.887	0.676	1.190	0.922	0.911	0.694	1.209	0.92
0.94	0.845	0.829	0.606	1.157	0.860	0.852	0.622	1.175	0.871	0.875	0.641	1.191	0.884	0.900	0.658	1.208	0.897	0.924	0.675	1.227	0.94
0.96	0.822	0.841	0.589	1.174	0.836	0.864	0.605	1.192	0.847	0.888	0.623	1.208	0.859	0.915	0.640	1.225	0.872	0.937	0.656	1.245	0.96
0.98	0.798	0.853	0.572	1.190	0.811	0.876	0.588	1.208	0.822	0.900	0.605	1.225	0.834	0.923	0.621	1.242	0.847	0.950	0.637	1.262	0.98
1.00	0.774	0.864	0.555	1.206	0.786	0.888	0.570	1.224	0.797	0.912	0.586	1.241	0.809	0.937	0.602	1.258	0.821	0.963	0.618	1.278	1.00

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CHAPTER VII

MATHEMATICAL FORMULÆ

EXPONENTIAL AND LOGARITHMIC FORMULÆ

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

$$a^x = 1 + \log_\epsilon a + \frac{(x \log_\epsilon a)^2}{2!} + \frac{(x \log_\epsilon a)^3}{3!} + \cdots.$$

$$\epsilon^{jx} = 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \cos x + j \sin x.$$

$$\epsilon^{-jx} = \cos x - j \sin x.$$

$$\epsilon^{jx} + \epsilon^{-jx} = 2 \cos x.$$

$$\epsilon^{jx} - \epsilon^{-jx} = 2j \sin x.$$

$$\epsilon^x = \cosh x + \sinh x.$$

$$\epsilon^{-x} = \cosh x - \sinh x.$$

$$a^{x+y} = a^x a^y.$$

$$a^{x-y} = a^x a^{-y} = \frac{a^x}{a^y}.$$

If $x = a^u$, then $u = \log_a x$.

If $x = \epsilon^u$, then $u = \log_\epsilon x$.

$$\log(xy) = \log x + \log y.$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y.$$

$$\log(x^n) = n \log x.$$

$$\log\left(x^{\frac{1}{n}}\right) = \frac{1}{n} \log x.$$

$$\log_{10} x = \log_\epsilon \times \log_{10} \epsilon.$$

$$\log_\epsilon x = \log_{10} x \times \log_\epsilon 10.$$

$$\log_{10} \epsilon = 0.434294.$$

$$\log_\epsilon 10 = 2.302585.$$

$$\log_{10} \epsilon \times \log_\epsilon 10 = 1.$$

$$\epsilon = 2.718282.$$

$$\log_e x = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right\} [x > 0].$$

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \quad [x^2 < 1].$$

TRIGONOMETRIC FORMULÆ

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

$$\operatorname{ctn}(x \pm y) = \frac{\operatorname{ctn} x \operatorname{ctn} y \mp 1}{\operatorname{ctn} x \pm \operatorname{ctn} y}.$$

$$\sin x \pm \sin y = 2\sin\tfrac{1}{2}(x \pm y)\cos\tfrac{1}{2}(x \mp y).$$

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x + y)\cos\tfrac{1}{2}(x - y).$$

$$\cos x - \cos y = -2\sin\tfrac{1}{2}(x + y)\sin\tfrac{1}{2}(x - y).$$

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cdot \cos y}.$$

$$\operatorname{ctn} x \pm \operatorname{ctn} y = \frac{\pm\sin(x \pm y)}{\sin x \cdot \sin y}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

$$\sin x = -\tfrac{1}{2}j\left(e^{xj} - e^{-xj}\right).$$

$$\cos x = \tfrac{1}{2}\left(e^{xj} + e^{-xj}\right).$$

$$\tan x = -j\frac{e^{2xj} - 1}{e^{2xj} + 1}.$$

$$A\cos x \pm B\sin x = \sqrt{A^2 + B^2}\cos(x \mp \phi),$$

where $\tan\phi = \dfrac{B}{A}.$

$$A\sin x \pm B\cos x = \sqrt{A^2 + B^2}\sin(x \pm \phi),$$

where $\tan\phi = \dfrac{B}{A}.$

$$\sin^2 x - \sin^2 y = \sin(x + y)\sin(x - y).$$

$$\cos^2 x - \cos^2 y = -\sin(x + y)\sin(x - y).$$

$$\cos^2 x - \sin^2 y = \cos(x + y)\cos(x - y).$$

$$\cos^2 x - \sin^2 x = \cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

$$\sin 2x = 2\sin x\cos x.$$

$$(\cos x \pm j\sin x)^n = \cos nx \pm j\sin nx.$$

	$-x.$	$90° \pm x.$	$180° \pm x.$	$270° \pm x.$	$360° \pm x.$
sin....................	$-\sin x$	$+\cos x$	$\mp\sin x$	$-\cos x$	$\pm\sin x$
cos....................	$+\cos x$	$\mp\sin x$	$-\cos x$	$\pm\sin x$	$+\cos x$
tan....................	$-\tan x$	$\mp\text{ctn}\, x$	$\pm\tan x$	$\mp\text{ctn}\, x$	$\pm\tan x$
ctn....................	$-\text{ctn}\, x$	$\mp\tan x$	$\pm\text{ctn}\, x$	$\mp\tan x$	$\pm\text{ctn}\, x$
sec....................	$+\sec x$	$\mp\csc x$	$-\sec x$	$\pm\csc x$	$+\sec x$
cosc..................	$-\csc x$	$+\sec x$	$\mp\csc x$	$-\sec x$	$\pm\csc x$

HYPERBOLIC FORMULÆ

$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

$$\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y.$$

$$\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y.$$

$$\cosh (x - y) = \cosh x \cosh y - \sinh x \sinh y.$$

$$\tanh (x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

$$\tanh (x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}.$$

$$\cosh^2 x - \sinh^2 x = 1.$$

$$1 - \tanh^2 x = \text{sech}^2 x.$$

$$1 - \text{ctnh}^2 x = \text{csch}^2 x.$$

$$A \sinh x \pm B \cosh x = \sqrt{A^2 - B^2} \sinh (x \pm \phi),$$

where

$$\tanh \phi = \frac{B}{A}.$$

$$A \cosh x \pm B \sinh x = \sqrt{A^2 - B^2} \cosh (x \pm \phi),$$

where

$$\tanh \phi = \frac{B}{A}.$$

$$\sinh x = \tfrac{1}{2} (\epsilon^x - \epsilon^{-x}) = - \sinh (- x) = - j \sin (jx).$$

$$\cosh x = \tfrac{1}{2} (\epsilon^x + \epsilon^{-x}) = \cosh (- x) = \cos (jx).$$

$$\tanh x = \frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}} = - \tanh (- x).$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2} (x + y) \cosh \tfrac{1}{2} (x - y).$$

$$\sinh x - \sinh y = 2 \cosh \tfrac{1}{2} (x + y) \sinh \tfrac{1}{2} (x - y).$$

$$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2} (x + y) \cosh \tfrac{1}{2} (x - y).$$

$$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2} (x + y) \sinh \tfrac{1}{2} (x - y).$$

$$\cosh x + \sinh x = \epsilon^x.$$

$$\cosh x - \sinh x = \epsilon^{-x}.$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdot$$

$$\cosh (x + jy) = \cosh x \cos y + j \sinh x \sin y.$$

$$\sinh (x + jy) = \sinh x \cos y + j \cosh x \sin y.$$

$$\cosh (x \pm jy) = \cos (y \mp jx).$$

$$\sinh (x \pm jy) = \pm j \sin (y \mp jx).$$

$$\sin (x \pm jy) = \pm \sinh (y \mp jx).$$

$$\cos (x \pm jy) = \cosh (y \mp jx).$$

COMPLEX QUANTITIES

Any function of a complex quantity can be put in the following form:

$$f(a + jb) = A + jB.$$

Addition:

$$(a + jb) + (c + jd) = A + jB,$$
$$A = a + c,\quad B = b + d.$$

Subtraction:

$$(a + jb) - (c + jd) = A + jB,$$
$$A = a - c,\quad B = b - d.$$

Multiplication:

$$(a + jb)(c + jd) = A + jB,$$
$$A = ac - bd,\quad B = be + ad.$$

Division:

$$\frac{a + jb}{c + jd} = A + jB,$$
$$A = \frac{ac + bd}{c^2 + d^2},\quad B = \frac{bc - ad}{c^2 + d^2}.$$

Square root:

$$\sqrt{a + jb} = A + jB,$$
$$A = \sqrt{\frac{a \pm \sqrt{a^2 + b^2}}{2}},\quad B = \sqrt{\frac{-a \pm \sqrt{a^2 + b^2}}{2}}.$$
$$\sqrt{j} = \frac{1 + j}{2}.$$

Logarithm:

$$\log (a + jb) = A + jB,$$
$$A = \tfrac{1}{2} \log (a^2 + b^2),\quad B = \tan^{-1} \frac{b}{a}.$$

Exponential:

$$a\epsilon^{c+jd} = A + jB,$$

$$A = a\epsilon^{c} \cos d, \quad B = a\epsilon^{c} \sin d.$$

$$a\epsilon^{c-jd} = A + jB,$$

$$A = a\epsilon^{c} \cos d, \quad B = -a\epsilon^{c} \sin d,$$

$$A^2 + B^2 = a^2\epsilon^{2c}.$$

$$p\epsilon^{a+jb} \times q\epsilon^{c+jd} = pq\epsilon^{a+c+j(b+d)} = A + jB,$$

$$A = pq\epsilon^{a+c} \cos (b + d),$$

$$B = pq\epsilon^{a+c} \sin (b + d).$$

$$\frac{p\epsilon^{a+jb}}{q\epsilon^{c+jd}} = \frac{p}{q}\epsilon^{(a-c)+j(b-d)} = A + jB,$$

$$A = \frac{p}{q}\epsilon^{a-c} \cos (b - d),$$

$$B = \frac{p}{q}\epsilon^{a-c} \sin (b - d).$$

$$\frac{a + jb}{c + jd}(e + jf) = A + jB,$$

$$A = \frac{aec - bfc + afd + ebd}{c^2 + d^2}, \quad B = \frac{acf + ebc - aed + bfd}{c^2 + d^2},$$

$$A = a + jb = r(\cos\beta + j\sin\beta).$$
$$= r\epsilon^{j\beta},$$

where $r = \sqrt{a^2 + b^2}, \quad \tan\beta = \frac{b}{a}.$

$$(a + jb)^n = r^n\{\cos\beta + j\sin\beta)^n\} = r^n e^{j\beta n}.$$

$$\sqrt[n]{a + jb} = r^{\frac{1}{n}} e^{\frac{j\beta}{n}}.$$

Roots of unity:

$$\sqrt[2]{1} = +1, \quad -1,$$

$$\sqrt[3]{1} = +1, \quad \frac{-1 + j\sqrt{3}}{2}, \quad \frac{-1 - j\sqrt{3}}{2},$$

$$\sqrt[4]{1} = +1, \quad -1, \quad +j, \quad -j,$$

$$\sqrt[8]{1} = +1, \quad -1, \quad +j, \quad -j, \quad \frac{+1+j}{\sqrt{2}}, \quad \frac{+1-j}{\sqrt{2}}, \quad \frac{-1+j}{\sqrt{2}}, \quad \frac{-1-j}{\sqrt{2}},$$

$$\sqrt[n]{1} = \cos\frac{2\pi k}{n} + j\sin\frac{2\pi k}{n} = e^{\frac{2\pi jk}{n}} \quad (k = 0, 1, 2, \ldots n - 1).$$

MISCELLANEOUS FORMULÆ

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n!\,a^{n-k}b^k}{(n-k)!\,k!} + \cdots \qquad [b^2 < a^2].$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad [x^2 < 1].$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n-1)x^2}{2!} \mp \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad [x^2 < 1].$$

$$(1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots \qquad [x^2<1].$$

$$(1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \cdots \qquad [x^2 < 1].$$

$$(1 \pm x)^{\frac{1}{3}} = 1 \pm \frac{1}{3}x - \frac{1\cdot 2}{3\cdot 6}x^2 \pm \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3 - \frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4 \pm \cdots \qquad [x^2 < 1].$$

$$(1 \pm x)^{-\frac{1}{3}} = 1 \mp \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 \mp \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4 \mp \cdots \qquad [x^2 < 1].$$

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \cdots$$

MISCELLANEOUS FUNCTIONS

$$ber\,(x) = 1 - \frac{x^4}{2^2\cdot 4^2} + \frac{x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2} - \frac{x^{12}}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10^2\cdot 12^2} + \cdots,$$

$$bei\,(x) = \frac{x^2}{2!} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \frac{x^{10}}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10^2} - \cdots,$$

$$ber'(x) = \frac{d\,ber\,(x)}{dx} = -\frac{x^3}{2^2\cdot 4} + \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} - \cdots,$$

$$bei'(x) = \frac{d\,bei\,x}{dx} = \frac{x}{2} - \frac{x^5}{2^2\cdot 4^2\cdot 6} + \frac{x^9}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10} - \cdots.$$

For $x > 6$ we have the following approximate expressions.

$$ber\,(x) = \frac{\epsilon^{\alpha} \cos \beta}{\sqrt{2\pi x}},$$

$$bei\,(x) = \frac{\epsilon^{\alpha} \sin \beta}{\sqrt{2\pi x}},$$

where

$$\alpha = 0.707105\,x + 0.0884\,x^{-1} - 0.046\,x^{-3},$$

$$\beta = 0.707105\,x - 0.39270 - 0.0884\,x^{-1} - 0.0625\,x^{-2} - 0.046x^{-3}.$$

INTERPOLATION FORMULA

$$f(x+t) = f(x) + t\Delta_1 f(x) + \frac{t(t-1)}{1\cdot 2}\Delta_2 f(x) + \frac{t(t-1)(t-2)}{1\cdot 2\cdot 3}\Delta_3 f(x) + \cdots,$$

where

$$\Delta_1 f\,(x) = f\,(x+1) - f\,(x),$$

$$\Delta_2 f\,(x) = \Delta_1 f(x+1) - \Delta_1 f\,(x).$$

TABLE XXI

Table of Napierian Logarithms to Nine Decimal Places for Numbers from 1 to 100

1	0.000000000	36	3.583518938	71	4.262679877
2	0.693147181	37	3.610917913	72	4.276666119
3	1.098612289	38	3.637586160	73	4.290459441
4	1.386294361	39	3.663561646	74	4.304065093
5	1.609437912	40	3.688879454	75	4.317488114
6	1.791759469	41	3.713572067	76	4.330733340
7	1.945910149	42	3.737669618	77	4.343805422
8	2.079441542	43	3.761200116	78	4.356708827
9	2.197224577	44	3.784189634	79	4.369447852
10	2.302585093	45	3.806662490	80	4.382026635
11	2.397895273	46	3.828641396	81	4.394339155
12	2.484906650	47	3.850147602	82	4.406719247
13	2.564949357	48	3.871201011	83	4.418840608
14	2.639057330	49	3.891820298	84	4.430816799
15	2.708050201	50	3.912023005	85	4.442651256
16	2.772588722	51	3.931825633	86	4.454347296
17	2.833213344	52	3.951243719	87	4.465908119
18	2.890371758	53	3.970291914	88	4.477336814
19	2.944438979	54	3.988984047	89	4.488636370
20	2.995732274	55	4.007333185	90	4.499809670
21	3.044522438	56	4.025351691	91	4.510859507
22	3.091042453	57	4.043051268	92	4.521788577
23	3.135494216	58	4.060443011	93	4.532599493
24	3.178053830	59	4.077537444	94	4.543294782
25	3.218875825	60	4.094344562	95	4.553876892
26	3.258096538	61	4.110873864	96	4.564348191
27	3.295836866	62	4.127134385	97	4.574710979
28	3.332204510	63	4.143134726	98	4.584967479
29	3.367295830	64	4.158883083	99	4.595119850
30	3.401197382	65	4.174387270	100	4.605170186
31	3.433987204	66	4.189654742		
32	3.465735903	67	4.204692619		
33	3.496507561	68	4.219507705		
34	3.526360525	69	4.234106505		
35	3.555348061	70	4.248495242		

TABLE XXII

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
0.00	1.00000	1.00000	1.00000	0.00000	0.50	1.64872	0.60653	1.12763	0.52110
0.01	1.01005	0.99005	1.00005	0.01000	0.51	1.66529	0.60050	1.13289	0.53240
0.02	1.02021	0.98020	1.00020	0.02000	0.52	1.68203	0.59452	1.13827	0.54375
0.03	1.03045	0.97045	1.00045	0.03000	0.53	1.69893	0.58860	1.14377	0.55516
0.04	1.04081	0.96080	1.00080	0.04001	0.54	1.71601	0.58275	1.14938	0.56663
0.05	1.05127	0.95123	1.00125	0.05002	0.55	1.73325	0.57695	1.15510	0.57815
0.06	1.06184	0.94177	1.00180	0.06004	0.56	1.75067	0.57121	1.16094	0.58973
0.07	1.07251	0.93239	1.00245	0.07006	0.57	1.76827	0.56553	1.16690	0.60137
0.08	1.08329	0.92312	1.00320	0.08009	0.58	1.78604	0.55990	1.17297	0.61307
0.09	1.09417	0.91393	1.00405	0.09012	0.59	1.80399	0.55433	1.17916	0.62483
0.10	1.10517	0.90484	1.00500	0.10017	0.60	1.82212	0.54881	1.18547	0.63665
0.11	1.11628	0.89583	1.00606	0.11022	0.61	1.84043	0.54335	1.19189	0.64854
0.12	1.12750	0.88692	1.00721	0.12029	0.62	1.85893	0.53794	1.19844	0.66049
0.13	1.13883	0.87810	1.00846	0.13037	0.63	1.87761	0.53259	1.20510	0.67251
0.14	1.15027	0.86936	1.00982	0.14046	0.64	1.89648	0.52729	1.21189	0.68459
0.15	1.16183	0.86071	1.01127	0.15056	0.65	1.91554	0.52205	1.21879	0.69675
0.16	1.17351	0.85214	1.01283	0.16068	0.66	1.93479	0.51685	1.22582	0.70897
0.17	1.18530	0.84366	1.01448	0.17082	0.67	1.95424	0.51171	1.23297	0.72126
0.18	1.19722	0.83527	1.01624	0.18097	0.68	1.97388	0.50662	1.24025	0.73363
0.19	1.20925	0.82696	1.01810	0.19115	0.69	1.99372	0.50158	1.24765	0.74607
0.20	1.22140	0.81873	1.02007	0.20134	0.70	2.01375	0.49659	1.25517	0.75858
0.21	1.23368	0.81058	1.02213	0.21555	0.71	2.03399	0.49164	1.26282	0.77117
0.22	1.24608	0.80252	1.02430	0.22178	0.72	2.05443	0.48675	1.27059	0.78384
0.23	1.25860	0.79453	1.02657	0.23203	0.73	2.07508	0.48191	1.27850	0.79659
0.24	1.27125	0.78663	1.02894	0.24231	0.74	2.09594	0.47711	1.28652	0.80941
0.25	1.28403	0.77880	1.03141	0.25261	0.75	2.11700	0.47237	1.29468	0.82232
0.26	1.29693	0.77105	1.03399	0.26294	0.76	2.13828	0.46767	1.30297	0.83530
0.27	1.30996	0.76338	1.03667	0.27329	0.77	2.15977	0.46301	1.31139	0.84838
0.28	1.32313	0.75578	1.03946	0.28367	0.78	2.18147	0.45841	1.31994	0.86153
0.29	1.33643	0.74826	1.04235	0.29408	0.79	2.20340	0.45384	1.32862	0.87478
0.30	1.34986	0.74082	1.04534	0.30452	0.80	2.22554	0.44933	1.33743	0.88811
0.31	1.36343	0.73345	1.04844	0.31499	0.81	2.24791	0.44486	1.34638	0.90152
0.32	1.37713	0.72615	1.05164	0.32549	0.82	2.27050	0.44043	1.35547	0.91503
0.33	1.39097	0.71892	1.05495	0.33602	0.83	2.29332	0.43605	1.36468	0.92863
0.34	1.40495	0.71177	1.05836	0.34659	0.84	2.31637	0.43171	1.37404	0.94233
0.35	1.41907	0.70469	1.06188	0.35719	0.85	2.33965	0.42741	1.38353	0.95612
0.36	1.43333	0.69768	1.06550	0.36783	0.86	2.36316	0.42316	1.39316	0.97000
0.37	1.44773	0.69073	1.06923	0.37850	0.87	2.38691	0.41895	1.40293	0.98398
0.38	1.46228	0.68386	1.07307	0.38921	0.88	2.41090	0.41478	1.41284	0.99806
0.39	1.47698	0.67706	1.07702	0.39996	0.89	2.43513	0.41066	1.42289	1.01224
0.40	1.49182	0.67032	1.08107	0.41075	0.90	2.45960	0.40657	1.43309	1.02652
0.41	1.50682	0.66365	1.08523	0.42158	0.91	2.48432	0.40252	1.44342	1.04090
0.42	1.52196	0.65705	1.08950	0.43246	0.92	2.50929	0.39852	1.45390	1.05539
0.43	1.53726	0.65051	1.09388	0.44337	0.93	2.53451	0.39455	1.46453	1.06998
0.44	1.55271	0.64404	1.09837	0.45434	0.94	2.55998	0.39063	1.47530	1.08468
0.45	1.56831	0.63763	1.10297	0.46534	0.95	2.58571	0.38674	1.48623	1.09948
0.46	1.58407	0.63128	1.10768	0.47640	0.96	2.61170	0.38289	1.49729	1.11440
0.47	1.59999	0.62500	1.11250	0.48750	0.97	2.63794	0.37908	1.50851	1.12943
0.48	1.61607	0.61878	1.11743	0.49865	0.98	2.66446	0.37531	1.51988	1.14457
0.49	1.63232	0.61263	1.12247	0.50985	0.99	2.69123	0.37158	1.53141	1.15983
0.50	1.64872	0.60653	1.12763	0.52120	1.00	2.71828	0.36788	1.54308	1.17520

TABLE XXII — (*Continued*)

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
1.00	2.71828	0.36788	1.54308	1.17520	1.50	4.48169	0.22313	2.35241	2.12928
1.01	2.74560	0.36422	1.55491	1.19069	1.51	4.52673	0.22091	2.37382	2.15291
1.02	2.77319	0.36059	1.56689	1.20630	1.52	4.57223	0.21871	2.39547	2.17676
1.03	2.80107	0.35701	1.57904	1.22203	1.53	4.61818	0.21654	2.41736	2.20082
1.04	2.82922	0.35345	1.59134	1.23788	1.54	4.66459	0.21438	2.43949	2.22510
1.05	2.85765	0.34994	1.60379	1.25386	1.55	4.71147	0.21225	2.46186	2.24961
1.06	2.88637	0.34646	1.61641	1.26996	1.56	4.75882	0.21014	2.48448	2.27434
1.07	2.91538	0.34301	1.62919	1.28619	1.57	4.80655	0.20805	2.50735	2.29930
1.08	2.94468	0.33960	1.64214	1.30254	1.58	4.85496	0.20598	2.53047	2.32449
1.09	2.97427	0.33622	1.65525	1.31903	1.59	4.90375	0.20393	2.55384	2.34991
1.10	3.00417	0.33287	1.66852	1.33565	1.60	4.95303	0.20190	2.57746	2.37557
1.11	3.03436	0.32956	1.68196	1.35240	1.61	5.00281	0.19989	2.60135	2.40146
1.12	3.06485	0.32628	1.69557	1.36929	1.62	5.05309	0.19790	2.62549	2.42760
1.13	3.09566	0.32303	1.70934	1.38631	1.63	5.10387	0.19593	2.64990	2.45397
1.14	3.12677	0.31982	1.72329	1.40347	1.64	5.15517	0.19398	2.67457	2.48059
1.15	3.15819	0.31664	1.73741	1.42078	1.65	5.20698	0.19205	2.69951	2.50746
1.16	3.18993	0.31349	1.75171	1.43822	1.66	5.25931	0.19014	2.72472	2.53459
1.17	3.22199	0.31037	1.76618	1.45581	1.67	5.31217	0.18825	2.75021	2.56196
1.18	3.25437	0.30728	1.78083	1.47355	1.68	5.36556	0.18637	2.77596	2.58959
1.19	3.28708	0.30422	1.79565	1.49143	1.69	5.41948	0.18452	2.80200	2.61748
1.20	3.32012	0.30119	1.81066	1.50946	1.70	5.47395	0.18268	2.82832	2.64563
1.21	3.35348	0.29820	1.82584	1.52764	1.71	5.52896	0.18087	2.85491	2.67405
1.22	3.38719	0.29523	1.84121	1.54598	1.72	5.58453	0.17907	2.88180	2.70273
1.23	3.42123	0.29229	1.85676	1.56447	1.73	5.64065	0.17728	2.90897	2.73168
1.24	3.45561	0.28938	1.87250	1.58311	1.74	5.69734	0.17552	2.93643	2.76091
1.25	3.49034	0.28650	1.88842	1.60192	1.75	5.75460	0.17377	2.96419	2.79041
1.26	3.52542	0.28365	1.90454	1.62088	1.76	5.81244	0.17204	2.99224	2.82020
1.27	3.56085	0.28083	1.92084	1.64001	1.77	5.87085	0.17033	3.02059	2.85026
1.28	3.59664	0.27804	1.93734	1.65930	1.78	5.92986	0.16864	3.04925	2.88061
1.29	3.63279	0.27527	1.95403	1.67876	1.79	5.98945	0.16696	3.07821	2.91125
1.30	3.66930	0.27253	1.97091	1.69838	1.80	6.04965	0.16530	3.10747	2.94217
1.31	3.70617	0.26982	1.98800	1.71818	1.81	6.11045	0.16365	3.13705	2.97340
1.32	3.74342	0.26714	2.00528	1.73814	1.82	6.17186	0.16203	3.16694	3.00492
1.33	3.78104	0.26448	2.02276	1.75828	1.83	6.23389	0.16041	3.19715	3.03674
1.34	3.81904	0.26185	2.04044	1.77860	1.84	6.29654	0.15882	3.22768	3.06886
1.35	3.85743	0.25924	2.05833	1.79909	1.85	6.35982	0.15724	3.25853	3.10129
1.36	3.89619	0.25666	2.07643	1.81977	1.86	6.42374	0.15567	3.28970	3.13403
1.37	3.93535	0.25411	2.09473	1.84062	1.87	6.48830	0.15412	3.32121	3.16709
1.38	3.97490	0.25158	2.11324	1.86166	1.88	6.55350	0.15259	3.35305	3.20046
1.39	4.01485	0.24908	2.13196	1.88289	1.89	6.61937	0.15107	3.38522	3.23415
1.40	4.05520	0.24660	2.15090	1.90430	1.90	6.68589	0.14957	3.41773	3.26816
1.41	4.09596	0.24414	2.17005	1.92591	1.91	6.75309	0.14808	3.45058	3.30250
1.42	4.13712	0.24171	2.18942	1.94770	1.92	6.82096	0.14661	3.48378	3.33718
1.43	4.17870	0.23931	2.20900	1.96970	1.93	6.88951	0.14515	3.51733	3.37218
1.44	4.22070	0.23693	2.22881	1.99188	1.94	6.95875	0.14370	3.55123	3.40752
1.45	4.26311	0.23457	2.24884	2.01428	1.95	7.02869	0.14227	3.58548	3.44321
1.46	4.30596	0.23224	2.26910	2.03686	1.96	7.09933	0.14086	3.62009	3.47923
1.47	4.34924	0.22993	2.28958	2.05965	1.97	7.17068	0.13946	3.65507	3.51561
1.48	4.39295	0.22764	2.31029	2.08265	1.98	7.24274	0.13807	3.69041	3.55234
1.49	4.43710	0.22537	2.33123	2.10586	1.99	7.31533	0.13670	3.72611	3.58942
1.50	4.48169	0.22313	2.35241	2.12928	2.00	7.38906	0.13534	3.76220	3.62686

TABLE XXII — (*Continued*)

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
2.0	7.38906	0.13534	3.76220	3.62686	4.0	54.5982	0.01832	27.3082	27.2899
2.1	8.16617	0.12246	4.14431	4.02186	4.1	60.3403	0.01657	30.1784	30.1619
2.2	9.02501	0.11080	4.56791	4.45711	4.2	66.6863	0.01500	33.3507	33.3357
2.3	9.97418	0.10026	5.03722	4.93696	4.3	73.6998	0.01357	36.8567	36.8431
2.4	11.0232	0.09072	5.55695	5.46623	4.4	81.4509	0.01228	40.7316	40.7193
2.5	12.1825	0.08208	6.13229	6.05020	4.5	90.0171	0.01111	45.0141	45.0030
2.6	13.4637	0.07427	5.76900	6.69473	4.6	99.4843	0.01005	49.7472	49.7371
2.7	14.8797	0.06721	7.47347	7.40626	4.7	109.947	0.00910	54.9781	54.9690
2.8	16.4446	0.06081	8.25273	8.19192	4.8	121.510	0.00823	60.7593	60.7511
2.9	18.1741	0.05502	9.11458	9.05956	4.9	134.290	0.00745	67.1486	67.1412
3.0	20.0855	0.04979	10.0677	10.0179	5.0	148.413	0.00674	74.2099	74.2032
3.1	22.1980	0.04505	11.1215	11.0765	5.1	164.022	0.00610	82.0140	82.0079
3.2	24.5325	0.04076	12.2866	12.2459	5.2	181.272	0.00552	90.6388	90.6333
3.3	27.1126	0.03688	13.5747	13.5379	5.3	200.337	0.00499	100.171	100.167
3.4	29.9641	0.03337	14.9987	14.9654	5.4	221.406	0.00452	110.705	110.701
3.5	33.1155	0.03020	16.5728	16.5426	5.5	244.692	0.00409	122.348	122.344
3.6	36.5982	0.02732	18.3128	18.2855	5.6	270.426	0.00370	135.215	135.211
3.7	40.4473	0.02472	20.2360	20.2113	5.7	298.867	0.00335	149.435	149.432
3.8	44.7012	0.02237	22.3618	22.3394	5.8	330.300	0.00303	165.151	165.148
3.9	49.4024	0.02024	24.7113	24.6911	5.9	365.037	0.00274	182.520	182.517
4.0	54.5982	0.01832	27.3082	27.2899	6.0	403.429	0.00248	201.716	201.713

TABLE XXIII

Trigonometric Functions

x	Cos x	Sin x	x	Cos x	Sin x	x	cos x	Sin x
0.20	0.9801	0.1987	0.82	0.6822	0.7311	1.44	0.1304	0.9915
0.22	0.9759	0.2182	0.84	0.6675	0.7446	1.46	0.1106	0.9939
0.24	0.9713	0.2377	0.86	0.6524	0.7578	1.48	0.0907	0.9959
0.26	0.9664	0.2571	0.88	0.6371	0.7707	1.50	0.0707	0.9975
0.28	0.9611	0.2763	0.90	0.6216	0.7833	1.52	0.0508	0.9987
0.30	0.9553	0.2956	0.92	0.6059	0.7956	1.54	0.0308	0.9995
0.32	0.9492	0.3145	0.94	0.5898	0.8076	1.56	0.0108	0.9999
0.34	0.9428	0.3335	0.96	0.5735	0.8192	1.58	−0.0092	0.9999
0.36	0.9359	0.3522	0.98	0.5590	0.8305	1.60	−0.0292	0.9995
0.38	0.9287	0.3709	1.00	0.5403	0.8415	1.62	−0.0492	0.9988
0.40	0.9211	0.3894	1.02	0.5235	0.8521	1.64	−0.0692	0.9976
0.42	0.9131	0.4078	1.04	0.5062	0.8624	1.66	−0.0891	0.9960
0.44	0.9048	0.4259	1.06	0.4889	0.8724	1.68	−0.1090	0.9940
0.46	0.8961	0.4439	1.08	0.4713	0.8820	1.70	−0.1289	0.9917
0.48	0.8870	0.4618	1.10	0.4536	0.8912	1.72	−0.1486	0.9889
0.50	0.8776	0.4794	1.12	0.4357	0.9001	1.74	−0.1684	0.9857
0.52	0.8678	0.4963	1.14	0.4176	0.9086	1.76	−0.1881	0.9822
0.54	0.8577	0.5137	1.16	0.3993	0.9168	1.78	−0.2076	0.9782
0.56	0.8473	0.5312	1.18	0.3809	0.9246	1.80	−0.2272	0.9738
0.58	0.8365	0.5479	1.20	0.3624	0.9320	1.82	−0.2466	0.9691
0.60	0.8253	0.5646	1.22	0.3436	0.9391	1.84	−0.2660	0.9640
0.62	0.8138	0.5810	1.24	0.3248	0.9458	1.86	−0.2852	0.9585
0.64	0.8021	0.5972	1.26	0.3058	0.9521	1.88	−0.3043	0.9526
0.66	0.7900	0.6131	1.28	0.2867	0.9580	1.90	−0.3233	0.9463
0.68	0.7776	0.6288	1.30	0.2675	0.9636	1.92	−0.3422	0.9396
0.70	0.7648	0.6442	1.32	0.2482	0.9687	1.94	−0.3609	0.9326
0.72	0.7518	0.6594	1.34	0.2287	0.9735	1.96	−0.3795	0.9252
0.74	0.7384	0.6743	1.36	0.2093	0.9779	1.98	−0.3979	0.9174
0.76	0.7248	0.6889	1.38	0.1896	0.9817	2.00	−0.4162	0.9093
0.78	0.7109	0.7033	1.40	0.1700	0.9855			
0.80	0.6967	0.7174	1.42	0.1502	0.9887			

TABLE XXIV

Binomial Coefficients for Interpolation by Differences

(For use in interpolation formula)

t	Coefficients of		t	Coefficients of		t	Coefficients of		t	Coefficients of	
	Δ_2	Δ_3		Δ_2	Δ_3		Δ_2	Δ_3		Δ_2	Δ_3
.01	−.005	+.003	.26	−.096	+.056	.51	−.125	+.062	.76	−.091	+.038
.02	−.010	+.006	.27	−.099	+.057	.52	−.125	+.062	.77	−.089	+.036
.03	−.015	+.010	.28	−.101	+.058	.53	−.125	+.061	.78	−.086	+.035
.04	−.019	+.013	.29	−.103	+.059	.54	−.124	+.060	.79	−.083	+.033
.05	−.024	+.015	.30	−.105	+.060	.55	−.124	+.060	.80	−.080	+.032
.06	−.028	+.018	.31	−.107	+.060	.56	−.124	+.059	.81	−.077	+.031
.07	−.033	+.021	.32	−.109	+.061	.57	−.123	+.058	.82	−.074	+.029
.08	−.037	+.024	.33	−.111	+.062	.58	−.122	+.058	.83	−.071	+.028
.09	−.041	+.026	.34	−.112	+.062	.59	−.121	+.057	.84	−.067	+.026
.10	−.045	+.028	.35	−.114	+.063	.60	−.120	+.056	.85	−.064	+.024
.11	−.049	+.031	.36	−.115	+.063	.61	−.119	+.055	.86	−.060	+.023
.12	−.053	+.033	.37	−.117	+.063	.62	−.118	+.054	.87	−.057	+.021
.13	−.057	+.035	.38	−.118	+.064	.63	−.117	+.053	.88	−.053	+.020
.14	−.060	+.037	.39	−.119	+.064	.64	−.115	+.052	.89	−.049	+.018
.15	−.064	+.039	.40	−.120	+.064	.65	−.114	+.051	.90	−.045	+.016
.16	−.067	+.041	.41	−.121	+.064	.66	−.112	+.050	.91	−.041	+.015
.17	−.071	+.043	.42	−.122	+.064	.67	−.111	+.049	.92	−.037	+.013
.18	−.074	+.045	.43	−.123	+.064	.68	−.109	+.048	.93	−.033	+.012
.19	−.077	+.047	.44	−.123	+.064	.69	−.107	+.047	.94	−.028	+.010
.20	−.080	+.048	.45	−.124	+.064	.70	−.105	+.045	.95	−.024	+.008
.21	−.083	+.049	.46	−.124	+.064	.71	−.103	+.044	.96	−.019	+.007
.22	−.086	+.051	.47	−.125	+.064	.72	−.101	+.043	.97	−.015	+.005
.23	−.089	+.052	.48	−.125	+.063	.73	−.099	+.042	.98	−.010	+.003
.24	−.091	+.053	.49	−.125	+.063	.74	−.096	+.040	.99	−.005	+.002
.25	−.094	+.055	.50	−.125	+.063	.75	−.094	+.039	1.00	−.000	+.000

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